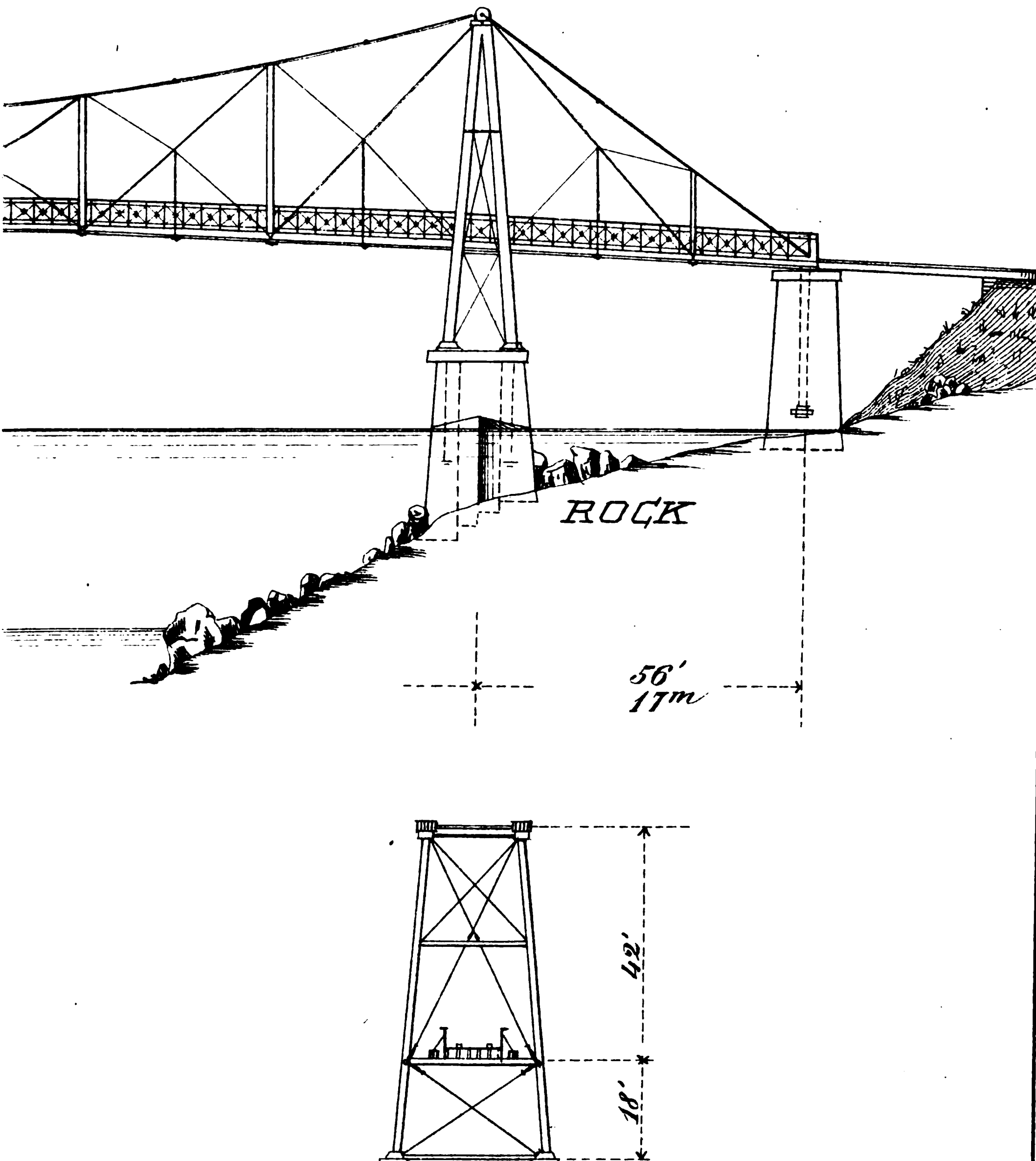

Principles of Economy in the Design of Metallic Bridges

Charles B. Bender

PRINCIPLES OF ECONOMY

IN THE

DESIGN OF METALLIC BRIDGES

BY

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PREFACE.

THE system of competitive design, combined with competitive prices, has produced in the United States the most economical and the most serviceable form of a single-span bridge. But there are other forms, such as arches, cantilever-trusses and -arches, and, for the very greatest spans, stiffened-wire suspension-bridges, of which the merits and proper proportions are less generally known, and outside of the United States the question as to the most economical form of truss is not yet everywhere settled.

In the book now presented the error often fallen into namely, of treating the web-system and the chords of single-span bridges separately or independently, has been avoided; and by multiplying the strain of each member of a fully loaded bridge of a fixed number of panels with the length of the member, and then adding the products, formulæ were obtained which agree with American practice, and with the author's own experience of some eighteen years.

Whilst examining in this way the more important forms of trusses, etc., it was considered well to add some historical notes as regards the origin of those forms or types. It will be noticed that this subject seems to have had special attraction to American and to German engineers, of whom

some, like the late J. A. Roebling and Mr. Albert Fink, developed their systems in the United States, while the late Mr. Bollman, of Baltimore, was the son of German parents.

The results of the book could not be arrived at without entering into mathematics, which for the most part will be found to be of an elementary nature. Only in a couple of the last paragraphs somewhat higher analysis was found to be unavoidable.

Though the author agrees with the majority of American engineers that the abuse of mathematical analysis—for instance, its application to the problems of earth-pressure, or of the best form and dimensions of rails, or of the strength of buckle-plates covered with ballast or gravel, and to similar problems—should not be continued, he nevertheless thinks that without the assistance of the queen of sciences the profession would soon arrive at exhaustion and would lose its proud position.

As a triumphant example of what the highest branches of the theory of elasticity, essentially the achievement of the genius of French savants, are capable of producing, the work done by the eminent Member of the Institute, Mr. M. G. Lamé, can be quoted. Armed with the highest analysis, he solved the difficult problem of the strains of hollow cylinders, and his formulæ, leading to radical changes in the construction of pieces of artillery, caused a complete revolution. While some twenty years ago it was not ventured to use initial velocities of projectiles of breach-loading cannon above 1100 feet, now velocities of 2000 feet are possible.

With such velocities projectiles shot from a 9½-inch gun weighing 18 tons, during the trials near Meppen in the year

1879, pierced armor 20 inches thick and continued their course unbroken for one and a quarter miles behind the target.

Such progress in its turn rendered the existing fleets obsolete, and caused new fleets with stronger armor and differently designed to be built—a process which is not yet ended.

In a similar degree the art of bridge-building owes as much to science, and always will owe as much, as it does to experiment. They must assist each other, and they must be cultivated with equal care.

This was one of the leading principles in preparing the book, which the author respectfully submits to the profession hoping that it will be kindly received and that it will lead to more complete labors in the same direction.

C. B. BENDER.

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PRINCIPLES OF ECONOMY IN THE DESIGN

OF

METALLIC BRIDGES.

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§ 1. The Sum of Strain-Lengths of all Members a Measure of Economy of Material.

An engineer designing a structure intends to mature such a plan that no part will be strained beyond previously specified maximum strains, and that the sum of expenses arising from all sources will be the minimum of present and future cost.

Economy may be secured by proper arrangement of the foundations, of the masonry, and of the superstructure, by reducing the cost of workmanship, of erection, and of the maintenance of the structure. Finally, the risk of execution and the loss of interest and of revenue during the time of construction have also to be considered.

Since it is impossible to form a mathematical expression embodying the different causes of expenditure, it will be necessary to develop for a special structure of considerable magnitude several projects, and to select the one best adapted, unless a specialist engineer already knows from previous study and from experience which general arrangement and what form will be the most appropriate for a given locality.

This book is intended to assist those who have to make

projects of great bridges. It treats of the theoretical quantities, and compares the more important types in this regard; whilst the remaining considerations of workmanship, erection, etc., are only touched upon where necessary. It will be found that those types of bridges which enjoy the advantage of theoretical minimum quantity of material, *properly* computed, are in reality also the best and most useful ones, or that practice and true theory are by no means in conflict.

The theoretical quantity of material required for a structure is proportional to the sum of the products obtained by multiplying the maximum strain of each member with its length. This sum we shall term "strain-length," and we denote it with "*Sl.*" In case of single-span trusses it is often sufficient to compute the strain-length under the supposition of full uniform load.

§ 2. Compression-Members.

If greater exactness were desired, the additional material given to long compression-members could be considered.

The formulæ used to determine this material are empirical, for the only theoretical one which in the last century was already given by Leonhard Euler, the originator of the common theory of the elastic line, does not even hold good for strains below the elastic limit.

A semi-empirical formula for the ultimate strength of compression-members is named after the late Professor Rankine, though it existed already earlier.* Also, this formula only approximately agrees with reality for the shortest as well as for the longest members.

* Indeed, Navier had already arrived at an expression which in its essence does not differ therefrom. Brix in Germany gave it in 1845; Professor Schwarz in the *Zeitschrift für Bauwesen*, Berlin, 1854, p. 518. Laisle and Schuebler's Book on Bridges (1857) contains the same formula (§§ 34 to 40 of this work). The French engineer Love derived a similar formula from Hodgkinson's experiments.

If I is the least moment of inertia of the sectional area of a compression-member, and S its sectional area, then $\sqrt{\frac{I}{S}}$ is termed the least radius of gyration, which we denote with r.

Further, if l is the length of the strut, then $\frac{l}{r} = n$ is the number indicating how many radii the strut is long.

Further, if s and c are empirical coefficients, the ultimate compressive strength, U, per square unit, according to Rankine, is expressed by

$$U = \frac{s}{1 + \frac{n^2}{c}}.$$

This formula is merely a combination of the formula for tension or for mere compression with that by Euler.

The quantities U and n being treated as variable, the formula represents a certain curve such as drawn on Plate I. It is seen that this curve for $n = 0$ begins with $U = S$, and that its tangent at this point stands at a right angle with the axis of the U.

The curve at first presents negative radii of curvature, then changes its curvature, and the values U decrease until they become 0 for infinitely long struts.

There is no explanation why there should be this reversion of curvature, or why U should slowly decrease for small values n. On the same plate there are laid down the results of numerous experiments; and other curves, which seem to conform tolerably well with those results, are laid down.

These curves start, not with an angle of 90 degrees, but with very small angles, $\frac{dn}{dU} = -\frac{c}{s}$.

The corresponding new formula, no less empirical, would be

$$U = \frac{S}{1 + \frac{n}{c}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where S and c are new constants, and $n = \frac{l}{r}$ as before.

The coefficient S is the *pure crushing* strength of the material, which for iron as well as for steel, contrary to the belief of Hodgkinson, but in agreement with many newer experiments (especially Bauschinger of Munich), was found *greater* than the ultimate tensile strength. The number c depends on the form of the member, on its manufacture, on the condition of its end-bearings, etc.

So far as present experiments go, it is greatest for riveted hollow columns composed of rolled channel segments, and this coefficient (according to experiments made by the engineers of the Cincinnati Southern Railroad) may be assumed to be 100 for flat-ended members, and to be 50 for posts with rounded ends.

These columns were originally invented by Mr. Schweikardi, about the year 1856, and were first manufactured at the works of Ars-sur-Moselle near Metz, and used in Paris, as stated by the inventor in his French patents.

The comparative strength of other forms is exhibited on Plate I.

All curves seem to agree as to the rapid decrease of the compressive strength near to the absolute crushing strength, or in the neighborhood of $n = 0$. Woehler was unable to crush pieces only three diameters long: they failed by flexure.

Unfortunately the results on Plate I. are a collection of experimental records furnished by different experimenters for different kinds of iron, and were obtained with different species of not always very reliable testing-machines.

Moreover, the mode of manufacture of the test-pieces,

much more important than is generally believed, most probably was not the same with these different specimens.

A complete series of systematically arranged experiments, both on the elastic behavior under pure pressure, and pressure combined with flexure, and on the ultimate strength of compression-members of various design and manufacture, executed by means of reliable lever testing-machines and with fine measuring apparatus, by a commission of scientific as well as experienced experimenters, is much to be desired.

It is known that the majority of plate-girders of iron as well as of steel, though of equal top- and bottom-flanges, break by tension (see Appendix No. 1).

But it does not follow that the same result must also be expected from skeleton-structures. A compression-member, as it were, is in a state of unstable equilibrium. If its middle line of pressure does not coincide with its geometrical axis, its ultimate resistance is much diminished, and pressure would have the effect of increasing any non-coincidence of these lines.

Bridge-members are sometimes made up hurriedly or in a cheap manner. Badly laid out holes; badly punched or, even if drilled, badly matching holes; badly fitting rivets; drift-pins used both in the shops and on the scaffold; joints cheaply or carelessly designed; eccentric intersection of the gravity-lines of the members of a bridge, more frequently of the wind-bracing; original unknown strains imparted to these; diagonal strains unevenly distributed over the eye-bars or over the riveted bands; non-parallel bearings of the struts at their ends; secondary strains—sometimes as high as 180 per cent of the primary strains—or non-central position of pins; the difference of moduli of elasticity of the pieces constituting compression-members; improper masonry-bearings; change of temperature, and other causes, contribute much to reduce the strength of such members.

If compression-members are composed of ribs connected by latticing or by lacing, the liability of crippling or bulging of the lattice-bars or of the ribs between the joint-points, or in case of ribs connected by thin plates the liability of their bulging, is first to be considered.

Then the single compression-members as limited by the joint-points of a bridge have to be considered as units capable of crippling.

And finally, a number of such chord-pieces composing a chord and being connected with another chord by lateral bracing form one great compression-member as long as the bridge—if a common truss-bridge or an arch—which must be looked upon as capable of crippling.

It is not impossible that under aggravated conditions such as enumerated a member may fail under strains caused by great working loads. The author knows at least of one bridge which has failed from such a cause, and he has learned that for similar reasons others—both in Europe and in America—were dismounted.

For the purpose of developing the principles of economic design it is sufficient to treat tension and pressure alike, for these reasons: With compression-members it has become the general practice not to subtract the area of the rivet-holes, and they can be joined together with less material than needed for tensional joints. The corrections for the stiffness of compression-members may be embraced by a coefficient of experience which combined with Sl gives the actual weight of the carrying-frame of a bridge.

Formula (1) was used by the author in practice since 1881. It has been confirmed, within the limits of practice,—that is, up to struts about 200 radii long,—by some 350 experiments made on iron and steel by James Christie, Esq., of Pencoyd (see Appendix No. 2 and Plates I. *a*, *b*, *c*). Mr. Christie's experiments fill a great part of the gap referred to and will be welcomed by the profession. We are obliged for his permission to utilize his results in this book.

§ 3. Tension-Members under Strains from Permanent and from Movable Loads.

Another consideration, of late abundantly and perhaps too much emphasized both in Germany and in bridge-specifications in the United States, is that of greater admissible strains for members bearing permanent stress, and correspondingly smaller strains per square unit for tensions caused by movable loads.

Though, naturally, long spans designed under such specifications become comparatively lighter than short spans, this consideration does not so much influence the *principles* of economic design as might be expected at a first glance.

It does not refer to compression-members. It refers principally to tension of those members which compose the floor, and also to the lighter parts of the web of a bridge. The strains of a member which has to carry pressure as well as tension are comparatively small, and the design can be so modified (mainly by arranging long panels) as to reduce either to nothing or at least to a comparatively small quantity the additional material considered necessary on account of changing strains.

Moreover, not only their value but also their *frequency*, and the effect of real *percussions** (such as caused by a derailed train) should receive proper consideration.

Besides, it must be remembered that the maximum movable loads for which structures are designed are not the rule, but of rare occurrence. The average or the *ordinary* loads should receive special attention. Similar considerations

* This was done by Gerber in Germany, who strained the iron for forces arising from permanent loads up to the elastic limit, and gave a factor of safety of 3, calculated on the elastic limit, for strains from movable loads. The Pauli bridge at Mainz was constructed on this principle in the year 1857. (See also Culmann's "Graphische Static," 1st edition, p. 401.)

refer to the wind-strains. According to an existing rule, the chords of fully loaded bridges must be proportioned to carry with the usual factor of safety pressures or tensions arising from the severest hurricanes. Such a specification, if applied to arches, leads to variations of areas which have no practical reason of existence.

In short, a mere empirical formula cannot be used without great modifications in special instances.

More especially as regards the fatigue of metal under repeated impacts, already the well-known English Iron Commission paid great attention to it, when also the question of increased strains caused by rapidly moving loads was to some extent considered.*

Later on, Cohn† in Austria showed by numerous experiments, consisting of repetitions—up to many millions—of blows upon iron bars of excellent quality, that the material *gradually* becomes of such a nature as to exhibit crystalline‡ structure in its fractures, obviously different from the fractures of the same material previous to the repeated blows. Cohn showed that these fractures exhibit more and more coarse-looking crystals the greater the number of blows, and that finally under such treatment the material loses all strength.

Then there is Fairbairn's well-known experiment with a riveted girder under repeated impacts.

Those made by Woehler in Germany must be considered as a continuation in another direction, namely, with a view

* Report of the Commissioners, etc., London, 1849, page 102: Cam experiments up to 100,000 repetitions on cast-iron, wrought-iron, and on an iron riveted tube. See also page 259 and page 470 of "Strength of Materials," by Thomas Cox, London, 1883.

† Karl Cohn in the *Zeitschrift des Oestreichischen Ingenieur Vereins*, 1851.

‡ Of course that obvious change of texture is the result of heavy impacts. If there are only repetitions of gentle vibrations, such as applied by Woehler, those coarse crystals are not produced.

towards finding rules for the proportions of railroad-axles and of railroad-car springs. His experiments, extending over a period longer than ten years, more especially refer to torsional oscillations of bars of cast-steel, of which we have no chemical analysis.

It is known that the subject of torsion of elastic bodies is far from being completely solved. Strains which are a combination of torsion, flexure, shearing, etc., of axles with projecting rings fixed into hubs, and of oscillations (modified by springs, as happened with Woehler's experiments), are but imperfectly known, especially at the breaking-point near the hub. It is therefore not so sure that other engineers calculating these strains would not arrive at results materially different from those given by Woehler. Certainly it is bold to use these few figures as the basis of far-reaching deductions towards rigid rules for the design of statical structures.

Director Woehler's experiments were continued by Professor Spangenberg, who found that the numbers of repetitions were not fully reliable, and that the results of experiments with Westphalian iron fell very much below the previous results obtained with specimens of patent shaft-iron.

Whereas Woehler thought a factor of safety of 2 applied to his figures of unlimited durability would be sufficient, Professor Launhard, who is the originator of the far-reaching deductions referred to, specifies a factor of safety of 3.

Extensive experiments as to the repetitions of pure tensions or pure compressions, or simple flexures of iron or of soft steel, such as used for bridge-work, or as regards bridge details,—for instance, riveted joints or eye-bars,—do not exist.

And Herr Woehler very prudently expressly stated that *special* experiments are necessary, so strongly suggested

by his own results and especially referred to by him in this very respect. Had it been possible to derive from his results a definite general rule, expressible by a formula, the experimenter, who is a scientific engineer, would undoubtedly have done so himself. It would have been sufficient to plot the final results, when any pronounced regularity at once must present itself to the eye. The author tried this as early as 1866,* when he convinced himself that thus far no such regularity can be found in the results. Nor did the subject receive much attention in Germany until, seven years later, Professor Launhard assumed that there must exist such a regularity, and what at most should have remained a personal conjecture or hypothesis, suggesting continued systematic experiments, was pronounced as a newly discovered "physical law."

Launhard thought that this law must be expressed by the much-favored parabola. He forced such a line on the results, for he assumed that the ultimate strength of the steel on which Woehler experimented breaks with 55 tons per square inch, while in reality it did not break after 35,600,000 repetitions of strains of 60 tons, and only broke under this strain after 34,000,000 repetitions in case of another experiment.

Launhard was followed by numerous others, each of whom brought a formula of his own.

The considerable number of these formulæ, representing so many opinions as to that supposed hypothetical law, is the surest proof that all rest upon mere conjectures.

So far as the available final results of Woehler's experiments go, it is even possible to assume a straight line; and if such a line were placed parallel to the axis of the abscissæ

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* Upon the publication of a great part of Woehler's experimental results in the *Zeitschrift für Bauwesen* in Berlin.

(see Plate II.), it would be found to lie not farther* from as many experimental results than does the Launhard-Weyhrauch parabola. But in this case the factor of safety would be the usual constant.

One of the formulæ (Launhard-Weyhrauch) is this:

$$\text{Area} = \frac{\text{Maximum}}{S\left(1 + 0.5\,\dfrac{\text{Minimum}}{\text{Maximum}}\right)},$$

where S is the basis-strain per square unit, Maximum means the greatest total strain of a member, and Minimum its smallest strain.

Approximately there may be put:

$$\text{Area} = \frac{\text{Max.}}{S}\left(1 - 0.5\,\frac{\text{Min.}}{\text{Max.}}\right) = \frac{\text{Max.}}{S} - \frac{\text{Min.}}{2S};\dagger$$

$$\text{Area} = \frac{\text{Strain from permanent load}}{2S} + \frac{\text{Strain from movable load}}{S}.$$

But this is a very old formula, used long ago in England and in the United States, also suggested at the time of

* The diagram shows that indeed of the three hypotheses, parabola, ellipse, and straight line, the ellipse gives the least sum of deviations from the actual results. The parabola gives the greatest sum of errors and the greatest sum of squares of errors, so that, as far as these results go, the straight line, or *constant* value (of 20 tons), is more probable than the parabola, and the ellipse more than either of the others.

† It has long been the rule of many engineers to add the absolute values of negative and positive strains occurring in a member, so that if there is a tension from permanent load, it has to be diminished by the pressure from movable load in order to arrive at the real starting or permanent tension.

the English Iron Commission, and, among others, by Rankine.*

It is based upon the consideration that if a movable load were imposed instantly the strain must be double of that strain caused by the same load imposed gradually and so as to cause no vibrations.

If there exist, as asserted, certain limiting strains for or under which a certain class of material can stand a really infinite number of repetitions of strains, it would seem more logical that the maximum strain used in practice should simply be below the limiting point. This, in fact, is the case in practice; for the maximum strains specified are about one half or one third of the elastic limit of the material used.

That material can be broken by repetitions of strains smaller than the ultimate strength was known long ago. That wire breaks sooner if bent to and fro instead of being bent only in one direction and straightened again, also is no novelty: and there are not many practical engineers or ironmen who have not tried this simple experiment.†

What Woehler has shown is this: that material under a permanent strain, but bearing additional variations of strains, breaks sooner (can endure a smaller number of repetitions of such experiments) the greater the permanent strain.

Interesting and suggestive as his experiments are, with reference to statical structures they have only revived the notion of building with a greater factor of safety those parts

* P. W. Barlow, of the Iron Commission, whilst recommending the factor of safety of 4 for permanent loads, desired the factor of 6 for railway structures. See "Strength of Materials," by Thomas Cox, London, 1883.

† See Morin, "Résistance de Matériaux," Paris, 1862. On the changes of iron of axles, see the reports of Mr. Marcoux and Mr. C. Arnoux, both officers of artillery and directors of the institution of Post-Coaches. There the experiment with wire is quoted, and many of the conclusions at which Woehler arrived much later are drawn from actual practice.

of a bridge which have to suffer most from sudden movable loads, such as are accompanied with violent vibrations, or such members of structures which have to bear their maximum strains more frequently than others.

The precise factor of safety for tensile maximum strains, until there are more complete experimental results, remains as much subject to personal judgment as the factor of safety of long compression-members.

In the subsequent investigations a constant maximum strain which is always considerably smaller than the elastic limit of the material will be supposed, unless expressly stated otherwise.*

§ 4. The Theorem of Clapeyron.

This theorem is part of a general principle. An exterior force multiplied with the displacement in its direction of its point of application equals double the sum of all interior work of the body elastically deformed. (See Lamé, "Leçons sur la théorie mathématique de l'élasticité des corps solides," deuxième édition, Paris, 1866.)

From this principle applied to skeleton-structures it follows that if a load were placed upon a joint-point, the deflection of this point multiplied by the load would equal the sum of all products obtained by multiplying the extension or compression of each member with its force due to that load. All terms of this sum must be positive, because if a force is negative (pressure) the alteration of length of the member will also be negative (the member will be short-

* As an illustration of the gradual decrease of strength of metal, there may be mentioned that six cast-iron girders of the London and Brighton Railway cracked in December, 1883, and four such girders at the Denmark Hill Station in London, though unloaded, fell in May, 1884. In consequence thereof these cast-iron girders are now forbidden.

ened), and the product of two negative factors gives a positive quantity.

Let us denote with

E, the modulus of elasticity, supposed to be a constant weight per square unit;

$F_1, F_2, F_3, \ldots$, the total forces acting in the members
1, 2, 3, . . ., which have the sectional areas
$s_1, s_2, s_3, \ldots$, and which have the lengths
$l_1, l_2, l_3, \ldots$;

w, the sum of deflections, or the ways described by the equal panel-loads W.

The theorem of Clapeyron furnishes the formula

$$W.w = \frac{1}{E}\left(\frac{F_1^2.l_1}{s_1} + \frac{F_2^2.l_2}{s_2} + \frac{F_3^2.l_3}{s_3} + \ldots\right) \quad . \quad . \quad (2)$$

Now suppose the structure to be designed in such a manner that under the loads W all members are exactly strained to the same amount of strain, S, per square unit, so that

$$S = \frac{F_1}{s_1} = \frac{F_2}{s_2} = \frac{F_3}{s_3} = \ldots;$$

which introduced into formula (2) gives

$$\frac{W.w.E}{S^2} = s_1l_1 + s_2l_2 + s_3l_3 + \ldots = \text{Volume of structure.}$$

It is evident that the total strains or forces, F, acting in the various members, are certain functions of the panel-loads, W, and of the dimensions and angles of the structure, so that we may put

$$F_1 = f_1 . W, \quad F_2 = f_2 . W, \quad F_3 = f_3 . W, \quad \text{etc.,}$$

where $f_1, f_2, f_3, \ldots$ are the functions for the members 1, 2, 3, . . ., and contain the lengths and angles of the skeleton-figure of the girder, arch, etc.

It results that

$$s_1 = f_1 \cdot \frac{W}{S}, \quad s_2 = f_2 \cdot \frac{W}{S}, \quad s_3 = f_3 \cdot \frac{W}{S}, \text{ etc.}$$

If these values are substituted in the expression for the volume of the structure, there is obtained

$$w \cdot \frac{E}{S} = f_1 \cdot l_1 + f_2 \cdot l_2 + f_3 \cdot l_3 + \ldots,$$

where W has disappeared.

The left side of the equation contains w, or the sum of deflections of the panel-points multiplied by the constant factor $\frac{E}{S}$.

The right side is a function of lengths and of angles of the skeleton-figure of the structure, and it is capable of being made a minimum. And this being done, the sum of deflections will be a minimum, and also the volume of the structure will be a minimum.

Hence the most economical structure carrying the panel-loads W for a certain specified strain of all of its members has also the least sum of deflection.

In a structure of uniform strength throughout, the least volume, or the greatest economy of material, and the greatest stability coincide.

To find such variations of the dimensions and angles as will give a minimum for the sum of deflections of the equally and fully loaded points of a structure embodies the greater part of the problem of economic design, so far as the quantity of material is concerned.

§ 5. Primary and Secondary Strains.

In the subsequent investigations it is supposed that we are able to calculate the strains of statical structures with a sufficient approximation to reality.

The strains calculated under the supposition that plate-girders are homogeneous beams, or that the joint-points of skeleton-structures are mathematical hinges, are termed primary strains.

Those strains which arise from the fact that the joints are more or less rigid, or which are caused by the gravity-lines of the members of a structure not meeting in the mathematical joint-points, are termed secondary strains.

These strains are caused by flexures of the members which in the calculation of the primary strains were supposed to remain straight lines.

These moments of flexure may not amount to any considerable percentage of the moments of flexure of the whole structure, and yet they may cause considerable additional local strains.

The correct intersection of gravity-lines of the members can be secured, and this principle should not be neglected in the design of lateral and transverse wind-bracing or of the attachment of the floor-beams.

The secondary strains arising from rigid connections are unavoidable, and the question arises how great they may be, how they can be diminished, and how they must be provided for.

Of late this important subject has received much attention in Germany, and in the year 1878 Herr Manderla solved the problem scientifically in a prize essay presented to the Munich Polytechnic School (see *Allgemeine Bauzeitung*,

Vienna, June 1880; also a popular representation of this theory by the author in the *Railroad Gazette*, April 1884).

With the aid of Manderla's formulæ the secondary strains of numerous, though simple, structures have already been calculated.

Professor Fraenkel of Dresden (see *Civil-Ingenieur*, Volumes XXVIII. and XXIX.) has commenced to examine (with an instrument of his own construction, which also has been used by the Dutch engineers in testing the strains of the Nymwegen bridge) experimentally the secondary strains, and it may be stated "that the strains at various points were not at all found to agree with calculation" (see also "Handbuch der Ingenieurwissenschaften: Brueckenbau," Leipzig, 1880, where Professor Steiner has treated Manderla's method). The instrument invented by Herr Stromeyer, which is based on the colored rings discovered by Hooke, if not too sensitive would be specially well adapted to experiments on secondary strains.

We cannot enter into the subject, but we shall state the principles of the theory of secondary strains:

The more nearly the structure is designed to contain the minimum volume of material (see § 4), or, what is the same thing, the less the sum of deflections of its joint-points, the smaller the secondary strains must be.

Hence the good rule to use the greatest practicable depth of truss.

The more evenly the total angles of the joints are deformed the more evenly the secondary strains must be distributed. For this reason parabolic girders would be preferable were it not that the average depths of such structures are smaller than the practically admissible depths of girders with parallel chords.

The longer the distances between connecting points the smaller the secondary strains will be.

Hence the good rule to use long panels and not to shorten artificially the members by interriveting the web-members where they cross each other.

The more nearly the tensile members are made to resemble mere flexible strings the more easily can they be bent without great strains; the less, therefore, the flexures of the compression-members will be. The flexures of these members are desired to be as small as possible, because they have great moments of inertia or are very stiff, and hence would receive great strains. And since these members must be safe against crippling, flexures would be more dangerous to them than to tensional members. The narrowness of ties, however, has a certain limit below which their own secondary strains again increase.

The practice of using eye-bars is advantageous as regards reduction of secondary strains, also because at the joints eye-bars are stronger against flexure than in their shanks, and much more so than broad, thin riveted ties. Eye-bars are attached in their gravity-lines, whilst this is not the case with the angles serving as diagonals of lattice-bridges.

It is good practice to build the end-posts and compressional chords of trusses as continuous unhinged members, for otherwise the pins will receive not inconsiderable torsional moments causing additional strains, and because nothing is gained by hinging those members together. The pins, when the bridge is once freed from the false-works, do not admit of rotation, because the secondary moments in a properly designed structure are not strong enough to overcome friction.

The secondary strains of pin-jointed structures arise only from movable loads.

Also, a part of the secondary strains of riveted structures may be assumed to have vanished by the settling of the structures during the removal of the false-works or under

the test-loads. But if the rivets are very numerous and are well driven at the joints of such structures, the head-friction may be sufficient to keep the joints rigid.

The author has calculated the secondary strains of a 100-foot Whipple truss, 20 feet deep, with panels 20 feet long. The maximum secondary strain was 8 per cent* of the admissible pressure of the top-chords near the centre. These members could easily have been reinforced by using sufficiently long and strong joint-plates. The secondary strains of the eye-bars were quite insignificant.

The secondary strains of riveted structures were calculated to be much greater. For triangular girders 32 to 100 per cent, for quadrangular 10 to 24 per cent in the top-chords were found.

Of a triangular, all pin-jointed girder, of which the tensile members are built of broad flats with eyes riveted thereto, secondary strains up to 66 and even 172 per cent were calculated at some points. This bridge of 118 feet span consists of 9 panels, it is 12.5 feet deep, and was built in South Germany.

For continuous girders secondary strains as high as 180 per cent have been calculated over the middle piers.

It is evident that the idea of gaining stiffness by using shapes of considerable moments of inertia for the tensile as well as for the compression members, and of using only moderately long or short panels, and the mode of interriveting diagonals at the points where the diagonals and posts cross each other, is utterly erroneous. Such structures have only an apparent but wholly deceptive stiffness. They are not strong, for the members are more taxed by flexures than by their primary strains. The calculation of strains of such structures should invariably be extended to the second-

* Only 640 pounds per square inch, or 6 per cent of the tensile maximum.

ary strains, for the primary strains as resulting from the application of the common and elementary formulæ are untrustworthy.

On the contrary, deep, long-panel, pin-jointed structures with eye-bars as tensional members are almost entirely free from secondary strains. They are the best and the most economical structures, provided that the principle of central intersection of gravity-lines is not only applied to the main girders, but also to the lateral and transverse bracing and to the attachment of the floor. Without this condition being fulfilled or, at least, duly considered, they lack more or less the lateral stiffness required.

§ 6. Redundant Members.

A statical skeleton-structure of which the strains of equilibrium can no longer be calculated with the six equations for the equilibrium of solid bodies has redundant members.

The members which connect two contiguous spans of a continuous girder, or the members which connect the halves of an arch into a stiff rib, or which brace the arch rigidly against the abutments, are redundant members.

As a general rule, redundant members lead to an increase of material and labor.

The strains of structures with redundant members in a high degree depend on original or accidental strains imparted to members during the process of erection. In fact, there is no certainty that the members will bear the strains for which they are proportioned.

The counter-rods of trusses if adjusted with original tensions must be considered as redundant members; hence it is necessary to make all diagonals in panels with counter-rods stronger, and it is advisable to be careful as regards adjustment.

Supposing a uniform known modulus of elasticity and perfect adjustment, it is possible to calculate the strains of a structure with any number of redundant members for a given mode of loading. But the calculation is often very tedious, and the maximum strain in each member can be found only by repeating the same calculation for a load on each separate panel-point.

There exist several modes of calculating the strains of structures with redundant members.

The first, and a very clear and simple one, is that by Herr O. Henrici, in his "Skeleton Structures."

A second mode, worked out by the author, consists in calculating the alterations of angles, divided by a web-diagonal, first by calculating from the proper triangle the alteration of the whole angle, and again by calculating the alteration of each of the two parts of the angle from the respective other triangles. The sum of these two alterations equals the alteration of the whole angle. Thus for each redundant member an equation is found.

The third method has been founded on the theorem of Clapeyron, and was given by Professor Mohr of Hanover (*Zeitschrift des Architecten und Ingenieur Vereins zu Hannover*, 1874, 1881).

Structures with redundant members may become very dangerous if they are so complicated that the strains can no longer be calculated and therefore are not calculated at all. Such structures were the piers of the late Tay bridge.*

* Read the evidence of the court of inquiry into the Tay bridge disaster of December 28, 1879, a blue-book containing 19,919 questions and answers, and other matter most interesting and instructive.

It must be added that already before the Tay bridge was finished a hurricane on February 3, 1877, blew down two 245-foot spans, one girder of a 145-foot span, and an iron tower. (See *Good Words*, 1878, page 105.)

That it is possible, in spite of the great factors of safety in use, for a bridge

§7. The Trussed Beam. (Bollman Truss.)

The beam trussed with two iron rods and a kingpost was already known and used at the end of last century. Ludwig Wernwag introduced it in the United States in the year 1808.

$AB = 2a$ (of Fig. 1, Plate III.) is a beam supposed to consist of two parts, AD and DB, trussed with the ties AC and BC, and with the post $CD = y$, and supposed to have perfectly flexible hinges at $ABCD$. A and B are the points of support. The force, P, is supposed to act in D.

The distance of D from the centre, E, of AB being x, and $CD = y$, the strains multiplied by the lengths of their members give the

$$\text{Strain-length} = Sl = 2P\left(y + \frac{a^2 - x^2}{y}\right),$$

which will be a minimum for $y^2 = a^2 - x^2$, and the minimum itself will be

$$Sl \text{ min.} = 4P.y.$$

Each of the four parts (AB, BC, CD, AC) of the truss has the same strain-length, Py.

It results that for a minimum deflection CE must be made equal to the radius a, and that C must lie in the periphery of a circle, of which AB is the diameter.

If a number of loads were placed in different points of AB, similar systems, ACB, could be formed, and a whole bridge could be built in this manner. This was really done

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to break down was proved a year ago by the fall of a Swiss bridge in the valley of Westhall. When this bridge was to be opened with much rejoicing in the quiet locality, it was loaded with heavy wagons, and though these stood quiet for a couple of minutes, the structure gave way and a number of people lost their lives.

by the late Mr. Bollman of Baltimore. The depths, however, were made equally great, because the theoretically best arrangement requiring too long posts under no circumstances could have been executed.

If a uniform load, $2ap$, were spread over AB, and an innumerable number of truss-posts with diagonals were used, each P must be considered to be $p \,.\, dx$, and it follows at once that the minimum strain-length required equals double the surface of the circle over AB as diameter multiplied by p, or

$$Sl = 2pa^2 \,.\, 3.14 = 6.28pa^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A Bollman truss of uniform depth, h, and a distributed load, $2ap$, with an infinite number of systems, has

$$Sl = 4pa^2 \left(\frac{h}{a} + \frac{2a}{3h}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For different proportions of $\frac{2a}{h}$ the coefficients are obtained as follows:

$$\left.\begin{array}{lcccccc} \text{For } \frac{2a}{h} = & 2 & 4 & 6 & 8 & 10 & 12 \\ \times\; Sl = pa^2. & 6.66 & 7.33 & 9.33 & 11.66 & 14.33 & 16.66 \end{array}\right\} \quad (5)$$

The pressure of the top-chord of the Bollman truss is

$$\frac{2}{3}p \,.\, \frac{a^2}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the top-chord has great rigidity in proportion to the transverse strength required for the whole truss, and if there are no real hinges or knife-edges at the joint-points, the strains may be very different from those supposed in the above calculations, and the top-chord must be treated as a continuous beam with yielding supports. The secondary strains

arising therefrom are so great that Manderla's theory would lead to erroneous results.

The beam trussed with a pair of rods and a kingpost is still frequently used in the United States, but we fear that the calculation of its strains has not often been correct.

In examining this combination an opportunity will be given of applying the elegant theorem of Clapeyron.*

Let E be the uniform modulus of elasticity of iron, I the moment of inertia of AB.

The post to have the length h; AD to be equal to DB.

The sectional areas of AB, AC, CD to be respectively $f_{,}, f_{,,}, f_{,,,}$; their lengths, $2a$, d, h.

The beam AB deflects at D to the amount δ. By this deflection a part, $p_{,}$, of p is neutralized, or

$$\delta = p_{,} \cdot \frac{5}{24} \cdot \frac{a^4}{EI}; \quad p_{,} = \frac{24E \,.\, I\delta}{5 \,.\, a^4}; \quad p_{,,} = p - p_{,};$$

and the pressure, x, in h is

$$x = \frac{10}{8} a(p - p_{,}).$$

$$\text{Pressure in } AB = -\frac{a\,.\,x}{2h}; \qquad \text{Compression} = -\frac{a^2 x}{h\,.\,f_{,}\,.\,E}.$$

$$\text{Tension in } d = \frac{d\,.\,x}{2h}; \qquad \text{Extension} = \frac{d^2 x}{h\,.\,f_{,,}\,.\,E}.$$

$$\text{Pressure in } h = -\frac{5a}{4}(p - p_{,}); \qquad \text{Compression} = -\frac{x\,.\,h}{f_{,,,}\,.\,E}.$$

* Professor Dr. Grashof in the second edition of his excellent "Theorie der Elasticitaet und Festigkeit," Berlin, 1878, § 124, has treated this combination in another manner. His example of a trussed *wooden* beam leads to the result that a real reinforcement is only possible if the truss is rather shallow.

Hence, by the theorem of Clapeyron,

$$\delta . x = \frac{x^2}{2E . h^2} . \left(\frac{a^3}{f_{,}} + \frac{d^3}{f_{,,}} + \frac{2h^3}{f_{,,,}}\right);$$

hence

$$\left.\begin{aligned}
p_{,} &= \frac{3I}{a^2 . h^2}(p - p_{,})\left(\frac{a^3}{f_{,}} + \frac{d^3}{f_{,,}} + \frac{2h^3}{f_{,,,}}\right); \\
\frac{p}{p_{,}} &= 1 + \frac{a^2 . h^2}{3I . \left(\frac{a^3}{f_{,}} + \frac{d^3}{f_{,,}} + \frac{2h^3}{f_{,,,}}\right)} = 1 + \varphi; \\
\frac{p}{p_{,,}} &= 1 + \frac{1}{\varphi}; \quad p_{,} = \frac{p}{1 + \varphi}; \quad p_{,,} = \frac{p . \varphi}{1 + \varphi}.
\end{aligned}\right\} \quad . \quad . \quad (7)$$

Or the maximum moment at D is found

$$= - \frac{a^2 . p}{8(1 + \varphi)}(\varphi - 4),$$

and the maximum positive moment at a point $\frac{3}{8}a$ from A is

$$\frac{a^2}{128}(9p_{,,} + 39p_{,}) = + \frac{a^2 . p}{8(1 + \varphi)} . \left(\frac{9}{16}\varphi + \frac{39}{16}\right).$$

In order to find the maximum strains of the beam, the greatest pressures arising from these moments must be added to the horizontal pressure per square unit, $\frac{5a^2 . p_{,,}}{8h . f_{,}}$.

Let an example be calculated:

Suppose $2a = 33'4'' = 400$ inches; the beam to be 18 inches deep, with $I = 1260$; area, $22''$. Hence

$$\frac{I}{9''} = 140 \text{ cub. ins.} = R.$$

Further, suppose the load per lineal inch to be 200 pounds.

Make $f_{,,,} = 6$; $h = 120$; $f_{,,} = 4$; $d = 233$.

Thereupon will be found $\varphi = 7.4$; $p_{,} = 23.8$; $p_{,,} = 176.2$.
The central moment will be -405000; Max. strain $= 2893$.
The positive max. moment, 785000; Max. strain $= 5610$.
The pressure from the truss-rods $= 1670$.

The maximum pressure in the beam therefore is 7280 pounds, whilst if the beam had been hinged in the centre the maximum pressure from flexure would have been 7143, and from the truss-rods there would have been 1510 pounds, giving a greatest pressure of 8653 pounds.

As regards the truss-rods, they would have been strained to 9720 pounds per square inch; whereas the supposition of a continuous beam gives 10640 pounds.

It results that in this instance, if the truss had been proportioned under the supposition of a hinged beam, whilst in reality it was continuous, the beam would have been 16 per cent too strong and the diagonals 10 per cent too light.

§ 8. Girders with Parallel Chords and Single Triangular Bracing. (Neville Truss.) Fig. 2, Plate III.

This style of truss was invented by the Belgian engineer Neville* in the year 1840, and built by him since 1846. Some time later it was improved by Mr. S. Whipple, and other

* Most probably it was suggested by I. Town's lattice-bridges (patented April 3, 1835). All web systems are simply suppressed but one. The English patent of Warren and Monzani is dated August 15, 1848. The details of this patent resemble much those of Neville, and are by no means as perfect as those of the Newark dike-bridge. The English patent of George Smart of 1822, relating to "mathematical chains," has a very remarkable specification, and in reality there is the lattice-bridge both for wood and for iron.

details were used by Warren-Monzani. In America it is called triangular truss, and was built by Whipple before 1850.

It is much more economical than the Bollman truss.

We suppose it to be composed of $2n$ panels, where n is an odd number. Each panel-point is supposed to carry a concentrated load, P.

The strain-lengths are easily summed up as follows: *

For the top-chords, $P.\frac{d^2}{h}.\frac{2n}{3}(n^2-1)$;

For the bottom-chords, $P.\frac{d^2}{h}.\frac{n}{3}(2n^2+1)$;

Together, $P.\frac{d^2}{h}.\frac{n}{3}(4n^2-1)$.

For the web, $P.\left(n^2.\frac{h^2+d^2}{h}+nh\right)$;

and for the whole bridge,

$$Sl = n.P.\left[d^2.\frac{4n^2+3n-1}{3h}+(n+1)h\right]; \quad . \quad . \quad (8)$$

which becomes a minimum of

$$2nP.h_{,}(n+1) = 2.\frac{n+1}{n}.\frac{h_{,}}{d}.p.a^2,$$

if

$$h_{,} = d\sqrt{\frac{4n^2+3n-1}{3(n+1)}}.$$

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* This easy operation is performed with the aid of the known formulæ

$$1+2+3+4+\ldots+(x-1)+x = \Sigma(x) = \frac{x.(x+1)}{1.2};$$

$$1^2+2^2+3^2+4^2+\ldots+(x-1)^2+x^2 = \Sigma(x^2) = \frac{x(x+1)(2x+1)}{1.2.3}.$$

For different n the following coefficients are obtained:

$$\left.\begin{array}{llllllll} \text{For} & 2n = & 2 & 6 & 10 & 14 & 18 & 22 \\ & \dfrac{h_{,}}{d} = & 1 & 1.91 & 2.52 & 3.00 & 3.41 & 3.78 \\ & \dfrac{2a}{h_{,}} = & 2 & 3.13 & 4 & 4.66 & 5.13 & 5.9 \\ & Sl \text{ min.} = pa^2 \times & 4 & 5.10 & 6.04 & 6.86 & 7.58 & 8.25 \end{array}\right\} (9)$$

It is seen that if the Neville truss shall give a minimum value of Sl,

*The depth must be greater than the panel-length.** *The greater the number of panels the shallower the corresponding best girder must be.*

The angles which the diagonals form with the horizontal line must be considerably greater than 60 degrees.

It is more important to reduce the chord-strains than to reduce the material of the web.

The next table contains the results of formula (9) if d is made $= h$ (angles of 45 degrees), and again if $h = d.\sqrt{3}$ (angles of 60 degrees).

$$\left.\begin{array}{llllllll} & 2n = & 2 & 6 & 10 & 14 & 18 & 22 \\ h = d, & Sl = pa^2 \times & 4 & 6.2 & 8.8 & 11.4 & 14.1 & 16.7 \\ h = d\sqrt{3}, & Sl = pa^2 \times & 4.6 & 5.13 & 6.46 & 7.91 & 9.40 & 10.91 \end{array}\right\} (10)$$

The beneficial effect of great depth is apparent. Notwithstanding that angles of 45 degrees would make the web light, by no means would such arrangement lead to a minimum of material for the whole girder.

Moreover, diagonals at 45 degrees would become very long,

* This property was first noticed to obtain for Pratt trusses by the author's former assistant, Mr. Emil Adler.

and would require much additional material in order to obtain the necessary stiffness under pressure.

A comparison of the coefficients of the Bollman truss with those of the Neville truss shows the great superiority of the latter. A Bollman truss built with a depth of one eighth of the span has the coefficient 11.66, whilst the Neville truss with fourteen panels, angles of 60 degrees, or about the same depth, furnishes the smaller coefficient of 7.9.

Applying the theorem of Clapeyron, the degrees of stiffness of these fully loaded girders stand in proportion to their strain-lengths. The Bollman truss, therefore, must also be inferior in this regard, and, owing to the different systems having the top-chord in common, considerable irregularities of deflections and dangerous secondary strains under passing concentrated loads must be caused. Both on account of this inconvenience and by reason of its not being sufficiently economical it has justly been abandoned. At the time when these trusses were built in the United States it was the general custom to use cast-iron compression-members; and since for small spans these cast-iron compression-members had to be made considerably heavier than needed for mere scientific reasons, and since the top-chord of the ideal Bollman truss is strained equally throughout, the defects of the system were less felt at that time, whilst its great simplicity and its handsome appearance recommended it to many.

The Fink truss is a plain two-panel truss with independent sub-systems. It was invented by Mr. Albert Fink* in juxtaposition to the Bollman truss. In its reversed form it had already been used in roof-work, and—with the exception of the independence of the sub-systems—it is identical with the truss of the German Wiegmann (1838) or with the French Polonceau roof-truss (1840).

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* Patented in America May 9, 1854.

Supposing the same specified strains throughout as those supposed for the Bollman truss, and building the Fink truss with sixteen panels, depth $h = \frac{a}{4}$ for the primary and secondary systems, and depth $h = \frac{a}{8}$ and $\frac{a}{16}$ for the tertiary and fourth systems, the coefficient of Sl becomes 8.62, which figure stands nearer to the result of the triangular truss of fourteen panels $\left(\text{depth } \frac{a}{4}\right)$ than the Bollman truss. Besides, the Fink truss is less subject to excessive secondary strains than the other.

In conclusion, it may be noticed that while the maximum chord-strain of the Neville is nearly $\frac{pa^2}{2h}$, the top-chord of the Bollman is strained 50 per cent higher throughout its whole length, and this strain is about 100 per cent greater than the average top-chord of the Neville truss.

§9. The Whipple Truss. (Quadrangular Systems. Fig. 3.)

The modern quadrangular trusses, or girders with parallel chords and upright posts, were originated by American engineers. The wooden trusses of Colonel Long (1836), both for single spans and for continuous girders, were built with vertical tension-members. In the Howe truss* (1840) iron

* The stiffening railings of the suspension-bridge at Freiburg, built by Chaley in 1832, are Howe trusses. The Italian architect Palladio, who lived at the commencement of the seventeenth century, built a 3-panel wooden Whipple truss of 108 feet span over the torrent of Cismone. The panels were divided in couples of sub-panels, and the bracing thereof was triangular. (See "Manuel de l'ingénieur," par A. Debauve, 10^{me} fascicule). According to a Vienna book of 1830 (by Weiss, an architect), the English Major By invented the Howe truss. (See *Zeitschrift des Oestreichischen Ingenieur Vereins* of 1855. Drawings are

verticals were substituted for wood, and the details correspondingly modified. In the year 1843, T. Willis Pratt of Boston, Massachusetts, placed the struts vertically and arranged iron diagonal tension-members.

To Mr. S. Whipple of Albany, in the State of New York, the introduction of the double system and of the *inclined* end-post or end-diagonal of the Pratt truss is due.

Three-panel Whipple deck-bridges were built, before Whipple's invention, for railroads in Brandenburg and Saxony. They were made of cast- and wrought-iron (see Heinzerling, "Bruecken in Eisen," page 126). But the merits of this system were not appreciated in Germany at that time. Mr. S. Whipple, however, already in 1847 published a book on his bridges, which is of historic value so far as it relates to the proportions of depth to length of span, number of panels, proper angles of diagonals, etc. He used already depths of one seventh of the spans. His merit is the greater because at that time box-girders and huge plate-girders were made to receive unmerited attention, and he stood quite unsupported when in the same book he proved the Britannia tubular system to be much less economical than the Whipple truss, and when he likewise in the same book proposed precisely the same system of suspension-bridges which much later was executed by Ordish.*

Only very lately in Germany an approach was made to the Whipple truss with inclined end-posts, and in fact outside of the United States the perfection of this most simple, most serviceable, and very economical truss, which can be adapted

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added, and the only essential deviation from the Howe truss seems to be the arrangement of the timbers of the chords, one on top of the other instead of side by side.

* In his book the first attempts were made at finding the strains of bowstring, girders by a graphical method. Whipple also seems to have been the engineer who first abandoned the counter-rods except where really necessary.

to by far the greatest number of different localities, even yet is not sufficiently appreciated.

We shall see that the Whipple truss presents almost the maximum attainable economy of the various types of single-span trusses. For small openings, up to about 150 feet, the Pratt truss, but with Whipple's inclined end-posts, diagonals intersecting only one panel, seems to be the most appropriate structure.

For the Whipple truss of $2n$ panels, n being an odd number, we obtain the strain-lengths as follows:

For the top-chord, $P\frac{d^2}{h}\left(\frac{2n^3}{3} + n^2 - \frac{11n}{3} + 2\right)$;

For the bottom-chord, $P\frac{d^2}{h}\left(\frac{2n^3}{3} - n^2 + \frac{10n}{3} - 2\right)$;

For both chords, $P\frac{d^2}{h}\cdot\frac{n}{3}(4n^2 - 1)$:

which is exactly the same quantity as found for the Neville truss.

As regards the web,

The end-posts give $(2n - 1)\frac{h^2 + d^2}{h}.P$;

Two hangers, $2h.P$;

Posts, $\frac{(n-3)(n-2)}{2}h.P$;

Short diagonals, $(n - 1)\frac{d^2 + h^2}{h}.P$;

Long diagonals, $\frac{(n-1)(n-2)}{2}.\frac{4d^2 + h^2}{h}.P.$

For the whole web-system there is obtained

$$Ph(n^2+4-n)+P\frac{d^2}{h}(2n^2-3n+2);$$

and for the whole truss,

$$Sl = P.h(n^2-n+4)+P\frac{d^2}{h}\left(\frac{4n^3+6n^2-10n+6}{3}\right). \quad (11)$$

We find the minimum $2Ph_{,}(n^2-n+4)$

for $$h_{,} = d\sqrt{\left(\frac{4n^3+6n^2-10n+6}{3(n^2-n+4)}\right)};$$

and the table No. 12:

$2n =$	6	10	14	18	22	(12)
$\frac{h_{,}}{d} =$	2.14	2.9	3.4	3.81	4.13	
Sl min. $= p.d^2 \times$	4.75	5.57	6.38	7.15	7.92	
$\frac{\text{Depth}}{\text{Span}} =$	$\frac{1}{2.8}$	$\frac{1}{3.44}$	$\frac{1}{4.1}$	$\frac{1}{4.7}$	$\frac{1}{5.3}$	

These values confirm the law already stated in the preceding section, that the truss in order to be as light as possible must be much deeper than would be required to make the web a minimum.

This can be easily seen; for the web becomes a minimum if

$$h = d.\sqrt{\left(\frac{2n^2-3n+2}{(n^2+4-n)}\right)},$$

and this value is always less than $h_{,}$.

Also, with this truss it is more essential to reduce the chord-material than that of the web; and we find again that the truss must be made *shallower for a greater* number of panels than for a smaller number.

Usually the depth of the Whipple truss is made equal to two panel-lengths.

The posts of such a truss are as long as the compression-members of the web of a Neville truss, with diagonals at angles of 60 *degrees.*

We find for

$$\left.\begin{array}{llllll} 2n = & 6 & 10 & 14 & 18 & 22 \\ Sl = p \cdot a^2 \cdot \dfrac{2n^3+9n^2-11n+27}{3n^2} = & 4.78 & 5.96 & 7.32 & 8.71 & 10.08 \\ \dfrac{\text{Span}}{\text{Depth}} = & 3 & 5 & 7 & 9 & 11 \end{array}\right\} \quad (13)$$

There is no difficulty in finding the coefficients of Sl for any other number of panels. Instead of building up general formulæ it will be preferable to make a diagram for each number of panels, and to find the strains numerically.

An expression

$$Sl = P \cdot \left(b \cdot h + c \cdot \frac{d^2}{h}\right)$$

will be obtained, where b and c are algebraical expressions containing powers of the number n, and the minimum is always obtained for

$$h_{\prime} = \sqrt{\frac{b}{c}} \cdot d$$

In this manner, for trusses of triple, quadruple, etc., inter-

section of diagonals, with or without sub-systems, special numerical expressions for Sl may be found.

It will be noticed that $b \, . \, h_{,} = c \, . \, \frac{d^2}{h_{,}}$; or that the *minimum strain-lengths Sl* are obtained if *the vertical strain-lengths are equal to the horizontal strain-lengths*;* hence that the total strain-lengths will be double the values of either of these two equal quantities, or

$$Sl \text{ min.} = 2h_{,} \, . \, b \, . \, P = 2P \, . \, d \, . \, \sqrt{b \, . \, c}.$$

Practice does not go so far, but it is satisfied if the web-material equals the material of the chords; and even such depths as would be necessary to fulfil this condition are not reached in case of long spans, for the reason that the cost of manufacture and of handling heavy masses, the cost of transportation, the very important lateral and oblique stiffness to resist wind, the cost of scaffolding, of masonry, and of erection, must be considered.

The greater the spans the more imperative become these considerations.

Let the height $h_{,,}$ be found, so that the strain-length of the web of a Whipple truss be equal to the strain-length of its chords. The condition is

$$h_{,,}(n^2 + 4 - n) + \frac{d^2}{h_{,,}}(2n^2 - 3n + 2) = \frac{d^2}{h_{,,}}\left(\frac{4n^3}{3} - \frac{n}{3}\right);$$

* Each diagonal strain can be resolved into a horizontal and a vertical force. The horizontal force multiplied by the panel-length is the horizontal strain-length of the diagonal, and the vertical force multiplied by the height is the vertical strain-length of the diagonal. The sum of both is the strain-length of the diagonal.

or

$$h_{,,} = d \cdot \sqrt{\left(\frac{4n^3 - 6n^2 + 8n - 6}{3(n^2 - n + 4)}\right)};$$

$$Sl = 2P\frac{d^2}{h_{,,}}\left(\frac{4n^2 - n}{3}\right).$$

For $2n =$	6	10	14	18	22	
$h_{,,} = d \times$	1.55	2.31	2.86	3.30	3.7	
$Sl = p \cdot a^2 \times$	5.00	5.71	6.50	7.25	7.95	(14)
$\frac{\text{Span}}{\text{Depth}} =$	3.9	4.3	5	5.5	6	
Sl minimum (see 12),	4.75	5.57	6.38	7.15	7.92	

This table shows *how little the strain-lengths of Whipple trusses, such as are practicable, differ from the theoretical minima.*

§ 10. Comparison of the Neville and Whipple Trusses.

Table (13) in comparison with (10) shows the superiority of the Whipple truss, equal numbers of panels and *equal lengths of web-struts* being supposed.

The preference due to the Whipple truss will appear still greater if the actual weights are compared. Its top-chords are only half as long as those of the Neville; hence, according to formula (1), § 2, and also supposing equal radii of gyration, they require only half of the additional area against crippling as needed for the Neville truss.

Further, the minimum top-chord of the Whipple truss has the strain $\left(\frac{5n}{2} - 3\right)\frac{Pd^2}{h}$; whereas the triangular truss presents only $2(n - 1)\frac{P \cdot d^2}{h}$.

It follows that with the Whipple truss there is less variation between the minimum and the maximum chord-strain than with triangular trusses. In other words, it is easier to vary exactly the top-chord sections, and if the end-sections cannot be made exactly so small as required by calculation, the loss is less with the Whipple than with the Neville truss.

On the other hand, the strain-length of the top-chords of the Neville is less than that of the Whipple, and correspondingly this has lighter bottom-chords than the triangular truss.

The end-posts in both trusses are the same.

But the vertical position of the intermediate posts of the Whipple truss is a source of considerable advantage. It admits of substantial connection of the floor-beams with these posts, further of somewhat easier erection, and at the post-feet, in case that the bottom-chords are composed of eye-bars, a good attachment for the lateral diagonals can be made.

The Whipple truss has only two suspenders.

If its bottom-chords are designed as *stiff* members there will always be at hand substantial parts capable of receiving wind-braces, which can be attached in such a manner that the different centre-lines intersect in the scientific joint-points. In case of *linked* bottom-chords such intersection is not easily possible, and the eye-bars are not capable of annihilating any important moments of flexure arising from the wind-diagonals. It is therefore necessary to create for Neville trusses, which have n suspenders, as well as for Whipple trusses, as long as they are built with linked bottom-chords, *stiff suspenders* capable of taking up those *moments arising from the eccentric application of the forces acting in the lateral bracing.*

Bridges with linked bottom-chords and mere tensile hangers are not so rigid even under loads moving with moderate velocity as must be desired, and the strength of their lateral bracing is not at all sufficiently utilized in case of strong winds.

It was due to this want of lateral and also of oblique or transverse stiffness under wind and as regards fast-moving trains that pin-jointed bridges were, not unjustly, condemned.

The author therefore has abandoned the general American practice of using mere suspenders in pin-jointed bridges. They are replaced by stiff tie-posts capable of taking up the horizontal moments arising from the wind-bracing.

The floor-beams are thoroughly built *into compact* posts at every bottom-joint of a through-bridge; and in case of deck-bridges the panel-loads are placed centrally on the joints, and the transverse and horizontal wind-braces attached in such a manner that their gravity-lines pass through the mathematical joint-points. These improvements were introduced into practice three years ago, and the result went fully up to the expectation. These bridges naturally are very stiff vertically, in exact proportion to the economy in design or in agreement with Lamé's observation, and they are as rigid horizontally and obliquely as the stiffest riveted bridges, but they are free from a considerable amount of the additional secondary strains to which riveted bridges built hitherto have been subjected.

In case of great and deep Whipple spans additional stiff diagonals may be built from the foot of each of the suspender-posts to the centres of the end-posts, at the ends of which the wind-braces will find proper places for attachment.

The strains of the posts of a Whipple truss vary less than those of the Neville truss.

This is due to the long diagonals stretching over more than one panel. It is true that triangular trusses with double systems of web-posts and diagonals offer the same advantage, and also that of the shorter top-chord pieces. But since under progressing loads both trusses receive greater strains in some parts of the webs than under full load, and as only

counter-diagonals in the Whipple truss, but members capable of pressure as well as of tension in the Neville truss, have to be designed, it will be found that the Whipple has still a slight advantage.

Its posts are only strained by pressure and serve equally well whether the pressure comes from the main diagonals or from the counters. From this consideration and because in case of manifold web-members of triangular trusses the compression-diagonals will intersect each other and therefore will cause inconvenience both as regards design and as regards erection, also because triangular trusses show greater second ary strains than the Whipple, the latter is preferred.

The trusses with vertical posts which serve as posts only, and with diagonals which serve only as ties, offer the further advantage that the pins or the rivets by which the diagonals and the counter-rods are connected with the posts will not be acted upon in opposite directions, such as is the case with the central web-members of Neville trusses, especially if made too shallow. For these the rivets would have to be arranged in such a manner that the head-friction alone offers the same safety as the diagonals connected. And pins would have to fit tightly with increased bearing area, to prevent wearing.

Finally, it may be noticed that, though for equal heights theoretically the chord-material should be the same both for the Neville and for the Whipple truss, nevertheless for the whole truss the advantage will be found in favor of the Whipple truss.

For fourteen panels, depth one to seven, the coefficients are obtained as follows: 7.32 for the Whipple and 7.43 for the Neville single system, and also 7.43 for the Neville double system.

§ 11. Other Properties of the Whipple Truss.

Deck-bridges have heavier posts than through-bridges.

For a Whipple deck-truss the value $2Ph(n-1)$ must be added, so that

$$Sl = P.\frac{d^2}{h}\frac{4n^3+6n^2-10n+6}{3}+P.\,h\,(n^2+n+2);$$

or for $h = 2d$,

$$Sl = pa^2\left(\frac{2n}{3}+3+\frac{1}{3n}+\frac{15}{n^2}\right).$$

$2n =$	6	10	14	18	22
$Sl = pa^2$,	6.77	7.00	8.00	9.24	10.48
Surplus over through-truss,	41	14	9	6	4 per ct.

On account of necessary stiffness under wind, deck-bridges are built shallower than through-spans, but their erection is more economical than would happen with through-bridges at the same localities. Deck-bridges with vertical end-posts, or also with inclined end-posts in case of several spans, save a part of the masonry. Such bridges, if of small or moderate spans, are usually designed with rail-ties placed directly upon the chords, the material of which serves as stringers, so that less material is needed than would happen if cross-beams and separate stringers were used. But the girders or trusses of deck-bridges should not be less apart than about ten feet, for there must be safety and equilibrium in case a train were to run off the rails during the heaviest storm, and there must be enough room to handle a train in such condition. The Whipple truss suffers from a slight

deficiency. In a through-span, at the first top-chord joint two diagonals and three other members meet, and the strains in these two diagonals can only be exactly calculated by the use of the theory of redundant members.*

But since the error made in using the ordinary mode of calculation amounts to only about one half of a panel-load divided by the number of panels, it is very small and can be neglected.

The web of a quadrangular truss will become a minimum if in every panel $\frac{2h^2 + 4d^2}{2h \cdot d}$ is a minimum.

The condition for this minimum is $h = 1.414d$. But it is known that $h = 2d$ leads to the more important economy in the chords. At the ends of a truss, however, this value of h does not lead to the greatest economy. For in the two end-panels $h = d$ would furnish the smallest strain-length of their parts. Further, it is nearly always impossible to exactly vary the sections of the top-chords, though for the bottom-chords, supposing eye-bars, this can be done. It follows that the height at the first post might be made $d\sqrt{2}$.

The alteration should not consist in a reduction of depth, but it should mean increase of panel-lengths and increased depth in the centre of the truss.

This design will have another advantage. The posts cannot be so perfectly varied as not to be stronger in the centre of the bridge than really necessary. The lightest posts will have more area and more stiffness than needed, and the possibility is suggested to utilize this surplus strength towards making them *longer*. By this increase of height the

* The engineers of the Cincinnati Southern Railroad have used a short link from which the two diagonals start, so that their strains can be exactly calculated without the theory of redundant members. This improvement was first applied to the Kentucky River bridge.

chord-strains are reduced, the chord-sections are more nearly equalized, whilst the lateral stability is only slightly reduced.

In Germany and in Holland semi-parabolic trusses of great span have lately been built much deeper (up to one sixth of the spans) than usual in the United States with great spans, and panels up to 18 feet length have been used in imitation of American practice. The diagonals of these bridges stretching over several panels, the calculation of their strains is somewhat tedious, for the theory of redundant members must be used if the strains are to be ascertained with approximate correctness.

Whether these very deep bridges were really more economical than would have been lower trusses with parallel chords remains doubtful. For if a tier of false-works can be saved, somewhat lower trusses with parallel chords may be preferable for the sake of more economical erection.

We shall now examine whether any advantage can be secured by arranging a Whipple truss with parallel chords in such a manner as to make the height nearly $1.4d$.

Formula (11) furnishes coefficients of Sl as follows:

$$\left.\begin{array}{llll}
2\dfrac{nd}{h}, & 6 & 7 & 8 \\
\left\{\begin{array}{l} 2n, \\ Sl = pa^2 \times \end{array}\right. & \left\{\begin{array}{l} 8 \\ 6.33 \end{array}\right. & \left\{\begin{array}{l} 10 \\ 7.02 \end{array}\right. & \left\{\begin{array}{l} 12 \\ 7.63 \end{array}\right. \\
\left\{\begin{array}{l} \text{Regular Whipple truss, } 2n = \\ Sl = pa^2 \times \end{array}\right. & \left\{\begin{array}{l} 12 \\ 6.64 \end{array}\right. & \left\{\begin{array}{l} 14 \\ 7.52 \end{array}\right. & \left\{\begin{array}{l} 16 \\ 8.01 \end{array}\right.
\end{array}\right\} (16)$$

It is seen, therefore, that indeed an economy of strain-lengths of about 5 per cent is obtained by lengthening the panels. The number of panels being reduced causes a considerable reduction of constructive labor, and also a reduction of joint-plates, rivets, pins, etc.

There will be a smaller number of floor-beams, but the stringers will become deeper and heavier.

The coefficients of table (16) are the best we have obtained for Whipple trusses with diagonals reaching over two panels.

The writer claims to have been the engineer who introduced the modern long panels. As early as 1872 he used 19-foot panels for a span of 152 feet; in 1873 and 1874 he used 19-foot and over 21-fôot panels in spans of 200 to 64 feet. Long panels of great bridges on the Laves and Pauli systems were used before those years, because the form of those systems of bridges accidentally led to such design; but the principle underlying the arrangement of long-panel Whipple or Neville trusses was not known at that time.

§ 12. Bowstring and Similar Girders.

The bow is supposed to be of parabolic form; its depth, h; and its equation (Fig. 4, Plate III.) is

$$\frac{x^2}{a^2} = \frac{h-y}{h}.$$

It is known that for full uniform load the bottom-chord tension is $\frac{pa^2}{2h}$. The suspender-posts are strained with p per unit of length of the bridge. Under full load the diagonals receive no strains. The top-chord has the strain-length

$$2 \cdot \frac{pa^2}{2h} \int_0^a \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx = \frac{pa^2}{h}\left(1 + \frac{4}{3}\frac{h^2}{a^2}\right).$$

The strain-length of the suspenders is

$$\frac{4}{3} p \frac{a^2}{h} \cdot \left(\frac{h^2}{a}\right),$$

and that of the bottom-chord is

$$\frac{pa^2}{h};$$

so that these two latter values together again give as much as the top-chord, and for the whole bridge under full uniform load there is

$$Sl = 2pa^2\left(\frac{a}{h} + \frac{4}{3}\cdot\frac{h}{a}\right).$$

The value becomes a minimum for $h = \frac{a}{2}\sqrt{3} = 0.87a$, and the minimum is $4.62 . pa^2$.

Because the chords are so heavy and the web so light this best depth of the parabolic girder must be very much greater than was found necessary for Neville or Whipple trusses.

For different depths the values of Sl are:

For $\frac{2a}{h}$ =	2	3	4	5	6	7	8	9	10
$Sl = pa^2 \times$	4.66	4.77	5.33	6.06	6.89	7.76	8.66	9.6	10.3

For practicable depths such as 1 : 6 or 1 : 7, the Whipple truss gave the coefficients 6.33 and 7.02. And yet these figures already include the greatest part of the maximum strain-lengths of the web-system of a Whipple truss, while the coefficients of the bowstring contain neither posts nor diagonals.

The parabolic bowstring, being divided into a number of equally long panels with vertical posts, and with the tensional diagonals in each of the $2n$ panels, has properties as follows: At every passage of the movable load certain maximum strains of all the diagonals are caused, and the *horizontal projections* of these maximum diagonal strains are all equally

great and equal to the tension of the string as caused by the *full movable* load divided by the number of panels of the whole span.

If a diagram is drawn geometrically similar to the bowstring, with AB (Fig. 4) $= \frac{pa^2}{2h}$, the diagonals and posts of this diagram by their lengths represent the actual strains, provided the scale of the diagram is such that AB measures the strain of the string from full movable load.

To find the strain-length of the diagonals and posts, the length of each of these members is measured, taking the panel-length as unit. The numbers found are squared and added together. The sum will then be multiplied by

$$\frac{p_0}{4} \cdot \frac{a^2}{n^2 . h}$$

where p_0 is the movable load per unit of length of the bridge.

The result must be added to the strain-length found above minus $\frac{8}{3} ap_1h$ or $\frac{8}{3} ap_0h$ according as the permanent load, p_1, or the movable load, p_0, is the greater.

The result gives the strain-length sufficiently correct for the purpose of comparison.

The values differ for different proportions of permanent to movable load.

If the movable load is double the permanent load; further, if a 12-panel girder, depth one sixth of the span, is supposed, we obtain for the web $2.03p_0 . a^2$; and after deduction of $\frac{8}{9} p_0 . a^2$ there remains $1.14p_0a^2 = 0.76pa^2$, which added to $6.89pa^2$ gives the strain-length of $7.65pa^2$, against only 6.65 to $6.33pa^2$ of the Whipple truss of the same depth. A Whipple truss of twelve panels, with the depth of only one eighth

instead of one sixth of the span, would have given only $7.63pa^2$.

Seeing that the bowstring presents so little economy of material, in addition to its more expensive form and to its inferior rigidity (against crippling of the top-chords or under strong winds), it is difficult to understand why only a few years ago great railway-bridges, arranged with double systems of diagonals, so intricate as regards reliable calculation of strains and as regards proper proportions, were erected on this plan in Europe.

There has been a tendency to tinker at the webs whilst true economy points to the opposite direction, namely, to simplification of design, and to saving in the chords by the adoption of trusses which shall be deep and vary in depth very little, if at all.

The parabolic bowstring was one of the first designs of trusses executed in iron.

Hoffmann built the first iron-girder bridge of this class in the year 1837 in Hungary.

In the year 1834 the Hanoverian engineer Laves sent a model of his lenticular truss, combining the arch and the suspension-principle, to the Anglo-French engineer Brunel. He published a description of his invention in 1835 in Havre in the French language, and he built many such trusses after 1839, both of wood and of iron. Brunel's bridges, especially the Saltash bridge, bear traces of the influence of Laves.

Later on the Bavarian engineer Von Pauli started out with the idea of building girders of which the neutral line is straight, in which line also the points of support are placed. In this manner he thought to remove longitudinal vibrations. This form, again, conducted him to the form given by Laves, and he modified the curvatures by making the strains in the chords of constant value in case the girder were fully loaded.

Finally, Schwedler of Berlin investigated a truss with all diagonals in tension, and without counter-diagonals. This investigation led him to the form of girder which carries his name, but which still has a few counter-diagonals near the centre.

None of these forms is of pronounced practical value. It is, however, good that they were once thoroughly examined, and that we now know that greater economy than is offered by the Whipple truss can hardly be obtained.

We have seen that the principle which the late Professor Culmann, the originator of graphical statics, has already pronounced in the first edition of his work, namely, that equidistant chords lead to the greatest economy, is almost absolutely true.

And having arrived at this result, it will be well not to continue to devote so much valuable time to special investigations of those and similar forms.

Let bridge-engineers and students rather direct their attention to subjects new and of greater usefulness. Of such subjects, as in all branches of engineering, there is an ever-increasing abundance.

§ 13. The Roof-Truss.

The form of truss exhibited on Plate III. (Fig. 5) possesses a number of interesting qualities, for which reason we shall shortly examine it.

A load, P, on the mth panel-point, counted from the nearest end, A, causes neither any diagonal strain nor any post-strain between the load itself and point A, nor any diagonal or any post strain in the half-truss BC. The diagonals of Fig. 5 would receive only tensions. It follows that a diagonal of this truss is in its maximum strain when all joint-points are fully loaded from the lower joint-point of this diagonal to the

nearest end-support. The loads on the other side of the diagonal have no influence on its strains. Each diagonal, therefore, is also in full strain when the whole truss is loaded.

If this truss is fully loaded, ***all parts do their maximum work.***

Those who expect economy from variation of depth or from reduction of web-strains * should be in favor of this truss, because the strain-length of the web is small. For such a structure of twelve panels and a depth of one sixth of the span only $1.92pa^2$ will be found for the web.

But all these as it would seem excellent qualities are secured at a great sacrifice, for the chords are very heavy.

If there are twelve panels, and the depth equals two panel-lengths, the strain of the end top-chord is $\frac{11}{7}$ of the central strain, and for the bottom-chord the proportion is as 11 to 6, so that the end-strains of the chords are nearly double the strains in the centre.

The strain-length for the top-chord is $4.93pa^2$, for the bottom-chord it is $4.25pa^2$, or for both together $9.18pa^2$; while the chords of a Whipple truss or of a Neville truss require only the material for a strain-length of $4pa^2$.

The total strain-length of the above truss would be $11.1pa^2$. For the Whipple it would only be $6.65pa^2$. That the coefficient 6.65 for the Whipple truss does not yet include the material for counter-strains in the web does not materially alter the result, because the lighter the web is theoretically, the greater will be the waste of material from practical reasons.

Roof-trusses, but with the diagonals in a direction opposite to that of Fig. 5, hence still less economical, are in use for drawbridges over the Rhine at Rotterdam. In the centres are the pivots.

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* The horizontal projection of the strain of any diagonal is $\frac{P \cdot a}{2h} = \frac{pa^2}{nh} =$ constant, which is exactly the same value as found for the parabolic truss, only that $p = p_1 + p_0$ is put for p_0.

Certainly this is not a good type of girder for a drawbridge, where the most economical form is desired, both in order to reduce the permanent weight and consequently friction, and in order to reduce the deflection to a minimum.

The secret of easily going turn-tables and drawbridges lies in extraordinarily great depths. But the depth must not be made great only at one point and then reduced to a minimum. It must be made as great as possible throughout; and in order to be able to adopt this plan legitimately it will be advisable to build single-track pivot-bridges *wider* than fixed spans.

§ 14. Continuous Girders and Cantilever-Bridges.

Girders built in one piece and resting on more than two supports, or continuous girders, do not lead to economy of material over properly proportioned single spans.

Only if enormous spans were made continuous, and if the points of reversion of flexure were fixed, by the introduction of hinges* or other interruptions of continuity, in such a manner as to make the moments of flexure over the middle piers rather intentionally great, but still within certain limits, some economy would be secured in case that single spans could not be built of sufficiently economic depths.

This economy is due to the great permanent load as compared with the movable load of great spans, to the concentration of greater masses near the middle piers (heavy chords and heavy webs are there combined), but principally because continuous girders with or without hinges, etc., have

* Such bridges were patented by C. de Bergue in England in the year 1865, by Gerber in Germany in 1866, and were reinvented early in 1867 by the author. Professor Ritter of Hanover treated another kind of continuous girders with hinges as early as 1862.

heavier webs than single spans but lighter chords than these, and thus lead to *lower economical trusses.*

The real continuous girders, of which the strains can only be calculated with consideration of the elasticity of the material and with acceptance of several doubtful or uncertain suppositions, are delicate; they are subject to disturbances by unequally high supports and by unequal heating by the sun; they require better masonry than single spans, and do not enjoy the advantage of the least number of parts, such as given by the inclined end-posts of the Whipple truss. They require more careful design at the chord-joints by reason of the great changes from pressure into tension, and conversely; they also require more careful erection than single spans, because of the changeable chord-strains and consequently of the necessity of many very well driven rivets. The webs of such continuous girders, unless the permanent load is very great, are heavier by just so much as is saved by their lighter chords.

The theory on which their calculation is founded neglects the deflections which are due to the web-members. This neglect may lead to considerable errors with the modern deep skeleton-trusses, though it was of little consequence at the time when plate-girders were used. The theory also rests on the supposition of a uniform modulus of elasticity of the finished members. And this modulus not only depends on the material, but also on design aud manufacture.

Bars and plates coming from the same mill are not of equal moduli. The collection of differing moduli* has received an additional indorsement by Professor Jenny of Vienna, who found moduli varying from 21,600,000 to 32,500,-

* "Practical Treatise on the Properties of Continuous Bridges," by C. B. Bender, 1876; also Mr. Christie's experiments prove the variability of moduli of elasticity.

000 pounds per square inch of the same kind of plate-iron, and moduli from 23,100,000 to 32,400,000 pounds for bar-iron.

Therefore these continuous girders have little to recommend them but the elegant mode of erection without false-works, by rolling them over the piers, first introduced by Benkisser of Pforzheim in Germany. And even this process is not used where, as in Switzerland and elsewhere, it was found to be inferior to erection on carefully prepared and also more extensive and more costly scaffolding.

We have records that the deflections of continuous skeleton-bridges under passing loads were found to be widely discrepant from the figures derived from the very theory on which they were designed; and this discrepancy is the most absolute proof that the suppositions of the theory in such instances were not realized.

Continuous girders also require great care as, if the reactions of the end-piers are too light and the ends are not secured to these piers, there is danger of their pivoting during a heavy storm.

Such disasters, as has been proved by experience, are of great extent in case of continuous bridges.

In Prussia and Holland continuous girders have not been used for a considerable time; and in the United States, where Colonel Long's wooden trusses were at first constructed as continuous beams, but were soon abandoned, several bridge-manufacturers and executive engineers have lately once more made careful estimates and arrived at the inevitable conclusion that there is no advantage in their use.

The application of the hinge to continuous girders soon followed its recommendation for the stiffening girders of suspension-bridges. And since that time a special and often-treated case related to the theory of continuity, namely, the single-span beam rigidly fixed at its ends, has been attended to.

The theory of the latter was first worked out by the French savant Girard at the end of last century, who used Euler's theory of the elastic line, which he translated into French.

In the year 1833 there appeared in the *American Mechanic's Magazine* the "tension-bridge" of M. A. Canfield of Paterson in the State of New Jersey (see Fig. 6, Plate II.).

This is unmistakably the beam rigidly fixed at its ends and with an independent middle span. This bridge was also reinvented by Sedley in England; and by Young (1865), who added the independent span of Canfield.

By the addition of this independent middle span the calculation of strains becomes more certain, the effects of changing temperatures can be avoided,* and yet the principle of the fixed beam, far from being sacrificed, is only more clearly expressed.

Such bridges are termed Cantilever-bridges. We define a cantilever-bridge as a bridge which consists of *one* opening, fixed with the minimum of cost at its ends, composed of two projecting arms and an independent middle span of convenient length.

It is obvious that the question of continuous girders can only arise if there is a *necessity* for *several large* spans. If only one very great span had to be built, and if an engineer were to construct other very great spans outside on dry land in order to confer upon the middle span the supposed benefit of continuity, he would not only waste his employer's funds, but would expose himself to well-merited ridicule.

Such design would be an impossibility where competitive designs and competitive prices are combined, or where the designs are entrusted to scientific special bridge-engineers.

* We do not include here the additional secondary strains caused by changes of temperature.

We shall now study the various forms of cantilevers and the principles which govern the more important modes of fixing their ends.

The parts which serve to fasten these ends are combined in the term "anchorage."

§ 15. Different Forms of Cantilevers. (Derrick- or Stay-Bridges.)

A, C, D of Fig. 7, Plate III., are supposed to be fixed points. At the end of $AB = a$ there acts a force P, which is supported by the tie *BD* and by the struts *AB* and *AD*. The strain-length of the system is

$$Sl = 2P\frac{a^2 + y^2}{y},$$

which becomes a minimum $= 4Pa$ for $y = a$.

For different proportions of $\frac{y}{a}$ other values of Sl are found, as follows:

$\frac{y}{a} =$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	(18)
$Sl = Pa \times$	5	4.33	4	5	6.66	8.5	10.4	12.33	14.3	

For $y = \frac{a}{3}$ the coefficient is still comparable with those of the best trusses known, but from there they rapidly decrease.

If a uniform load, pa, were spread over *AB*, a number of stays, *BD*, could be used in two ways, such as shown by Fig. 8 and by Fig. 9. Both designs are of old origin. That represented by Fig. 8 was proposed at the commencement of this century by the Frenchman Poyet.*

* Claude Marie Navier, "Rapport sur les ponts suspendus" (Paris, 1823).

Having adopted a certain height, h, we find (Fig. 8)

$$Sl = \frac{2p \cdot a}{n \cdot h}\left(nh^2 + \frac{a^2}{n^2}(1+4+9+\ldots+n^2)\right)$$

$$= 2\left(\frac{h}{a} + \frac{2n^2+3n+1}{6n^2} \cdot \frac{a}{h}\right)pa^2.$$

The minimum is $4\frac{h_{\prime}}{a} \cdot pa^2$ for $h_{\prime} = a\sqrt{\left(\frac{2n^2+3n+1}{6n^2}\right)}$.

If n is a great number, the minimum of Sl will be nearly $4\frac{pa^2}{\sqrt{3}} = 2.31pa^2$ for $h_{\prime} = 0.577a$.

For different ratios $\frac{a}{h}$ and different n we obtain:

$\frac{a}{h} =$	1	2	3	4	5	6	7	(19)
$n =$	2	4	6	8	10	12	14	
$Sl = pa^2 \times$	3.25	2.87	3.20	3.70	4.25	4.85	5.45	

For $n = \infty$ there is found

$Sl = pa^2 \times$	2.67	2.33	2.67	3.17	3.73	4.33	4.95

It was found that half a Whipple truss of fourteen panels, depth one seventh of span, furnishes the coefficient 3.66. The Poyet cantilever (of course without the anchorage) would give only $3.45pa^2$.

It has only one compression-member besides the post, the weight of which does not add to the permanent load to be carried outside of line AD.

But if P is the greatest possible concentrated panel-load—for instance, the driving-wheel load of a locomotive—each diagonal or stay must be designed to carry this great load.

Also arrangements are necessary to prevent the stays from sagging, and the bottom-chord must be considered as a continuous beam supported by elastically yielding piers.

The form represented by Fig. 9 suffers from the same defects in a still higher degree. For $h = \frac{2a}{7}$ and $n = 7$ it will be found that the strain-length is already $4.3pa^2$, and a number of separate anchorages would be required.

§ 16. Cantilevers designed as Trusses with Parallel Chords.

In constructing cantilevers as regular trusses with parallel chords, the principle is again observed that having decided upon the longest strut which shall be used, this strut must be placed in a vertical position so as to reduce the chord-strains rather than the strain-length of the web.

Wherever cantilevers are to be used, the spans are of such sizes that great depths are indispensable; hence manifold systems will have to be used in order to obtain favorable angles of diagonals and moderately long panels.

We consider first a cantilever with single system of web-members acted upon by a concentrated load, P, at the end. The number of panels may be n (Fig. 10).

For the chords there is found

$$P.\frac{d^2}{h}.n^2;$$

For the web, $$P.\frac{d^2 + 2h^2}{h}.n;$$

Together, $$Sl = Pa\left((n+1)\frac{d}{h} + 2.\frac{h}{d}\right). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For $h = d.\sqrt{\left(\frac{n+1}{2}\right)}$ the minimum is found

$$Sl \text{ min.} = 4Pa\sqrt{\left(\frac{n+1}{2}\right)}.$$

For different n we arrive at coefficients as follows:

$n =$	1	2	3	4	5	6	(21)
$\frac{h}{d} =$	1	1.22	1.41	1.58	1.73	1.87	
Sl min. $= Pa \times$	4	4.88	5.64	6.32	6.92	7.48	
And for $h = d$,							
$Sl\,(h = d) = Pa \times$	4	5	6	7	8	9	
of which web $= Pa \times$	3	3	3	3	3	3	

If these coefficients are compared with those of the Poyet cantilever for concentrated loads, it will be seen that the ordinary truss is equal to the latter for the proportions $\frac{h}{a} = 1$ and $\frac{h}{a} = 2$, but that it is better than the Poyet for longer arms.

We shall now examine a regular truss with a double system of diagonals, to serve as cantilever to carry uniform load, reference being had to Fig. 11, Plate III.

Each panel-load, K, is $\frac{pa}{n}$, n being supposed to be an even number.

There is first an ordinary single-intersection cantilever with $\frac{n}{2}$ panels, for which there is easily found for the chords

$$\frac{d^2.K}{h}.n.\frac{n^2+3n+2}{6};$$

For the web, $\frac{d^2.K}{h}.n.\frac{n+2}{2} + h.K\frac{n}{4}(n+2);$

Together,

$$Sl = \frac{d^2 . K}{6h} . n . (n + 2)(n + 4) + \frac{h . K}{4} . n . (n + 2)$$

$$= pa^2 . \frac{n+2}{2n}\left(\frac{d}{h} . \frac{n+4}{3} + \frac{h}{2d}\right).$$

For the other system, which also carries $\frac{n}{2}$ panel-loads, we have first again a cantilever of single intersection with $\frac{n-2}{2}$ panels, of which each is $2d = \frac{2a}{n}$ long, and for this is found

$$pa^2 . \frac{n-2}{2n}\left(\frac{d}{h} . \frac{n+2}{3} + \frac{h}{2d}\right).$$

The additional chord-strain in the last panel gives

$$K\frac{d^2}{h} n \frac{(n-2)}{2},$$

and the last diagonal lower chord and post, owing to the load $\frac{nK}{2}$, furnishes the value

$$Kdn \frac{d^2 + h^2}{h . d}.$$

These two last expressions added are

$$pa^2\left(\frac{d}{2h} + \frac{h}{n . d}\right).$$

The complete strain-length of the cantilever is

$$Sl = pa^2\left(\frac{2n^2 + 9n + 4}{6n} . \frac{d}{h} + \frac{n+2}{2n} . \frac{h}{d}\right), \quad . \quad . \quad . \quad (22)$$

which is a minimum for

$$h_{,} = d\sqrt{\left(\frac{2n^2 + 9n + 4}{3(n + 2)}\right)},$$

and the minimum itself is

$$pa^2\frac{n + 2}{n} \cdot \frac{h_{,}}{d}$$

$$\left.\begin{array}{llllll} \text{For } n = & 4 & 6 & 8 & 10 & 12 \\ \frac{h_{,}}{d} = & 2 & 2.33 & 2.61 & 2.86 & 3.09 \\ \frac{a}{h_{,}} = & 2 & 2.42 & 3.06 & 3.5 & 4 \\ Sl \text{ min.} = pa^2 \times & 3 & 3.13 & 3.26 & 3.43 & 3.60 \end{array}\right\} \quad (23)$$

Especially for $h = 2d$ there is found

$$\begin{array}{llllll} \frac{a}{h} = & 2 & 3 & 4 & 5 & 6 \\ Sl = pa^2 \times & 3 & 3.14 & 3.38 & 3.65 & 3.94 \end{array}$$

For the same proportions the Poyet cantilever has the coefficients 2.93 3.26 3.60 3.92 4.25

Thus it is seen that also for uniform load the regular truss is the form best adapted for a cantilever. It is not so much influenced by great single-panel loads as is the Poyet cantilever. It is also more regular in regard to deflections, its correct calculation of strains is simpler, and it is less influenced by secondary strains than the Poyet cantilever.

The slow increase of coefficients of the regular truss for increasing panel-numbers is due to the curve of moments being convex towards the origin of ordinates. The moments increase rapidly only near the origin.

It results from this property that the depth of a regular cantilever-truss may be made one third to one fourth of its length without materially increasing the strain-lengths from uniform load.

The average moment of flexure of a cantilever uniformly loaded is $\frac{pa^2}{6}$, whereas for a truss resting on two supports the average moment is $\frac{pa^2}{3}$. It would be a great error to judge that for this reason the cantilever weighs only half as much as does the truss on two supports. The cantilever has also to carry a concentrated load, P, at its end, and the average moment arising from this weight is exactly the same as the average moment of the ordinary truss of the length $2a$ and loaded with $2P$ in the middle. Also as regards the web-strains there is little difference. A Whipple truss with $h = \frac{a}{3} = 2d$ has the coefficient 6.64. For the same proportions (see 23) a pair of cantilever-trusses has 6.28. For a great number of panels the difference is more in favor of the cantilever.

Unfortunately the necessity of adding the anchorage destroys this advantage almost entirely.

§ 17. Cantilevers of Varying Depth.

It was seen that ordinary trusses of varying depths only in the rarest instances are a little more economical than those of uniform depth.

We shall now investigate the cantilever in a similar respect.

Let half of an ordinary roof-truss be considered.

The cantilever (Fig. 12, Pl. III.) is fixed at A and C; the depth is h.

The horizontal force in the chords at the mth panel-point from end B is

$$H = \frac{pa^2}{n^2} \cdot \frac{1 + 2 + 3 + 4 \ldots . + m}{\frac{m}{n} . h} = \frac{pa^2}{2h} \cdot \frac{m + 1}{n},$$

and the part of the shearing force which is absorbed by the inclined chord (or chords) at the same point is

$$H . \frac{h}{a} = \frac{pa}{2} \cdot \frac{m + 1}{n}.$$

The whole shearing force at the same point is $\frac{pa}{2} . \frac{2m}{n}$.

Hence the $\left(\frac{1}{2} + \frac{1}{2m}\right)$th part thereof is in the chords.

For a single concentrated load, P, at the end of the cantilever the whole force, P, remains in the chords.

For an infinite number of panels, finally half the shearing force is transferred by the chords and the other half by the web.

A little table may be formed as follows:

Number of panel-points.	0	1	2	3	4	5	6	7	8
Shearing force at panel-points................	1	2	3	4	5	6	7	8	9
Shearing force in chords..	1	1.5	2	2.5	3	3.5	4	4.5	5
Sums....................	1	2.5	4.5	7	10	13.5	17.5	22	27
Sums of whole shearing forces.................	1	3	6	10	15	12	28	36	45
Quotients...............	1	0.83	0.75	0.7	0.67	0.64	0.62	0 61	0.6
Rest in the web..........	0	0.17	0.25	0.3	0.33	0.36	0.38	0 39	0.40

But it must not be concluded that the last series of quotients compares with the real weights of the webs, because

the diagonals and posts of a cantilever designed as half a roof-truss do not act at favorable angles.

Moreover, the parts of the shearing forces which are transferred by the chords act at unfavorable angles, and the moments of flexure are not resisted by chords placed at favorable distances.

For an infinite number of panels the average force H for a truss with uniform depth is only two thirds of the average force H for a cantilever in shape of a roof-truss, and one of the chords (or both chords) being inclined, it has (or they have) a greater strain than H, and this chord is also longer than a.

Supposing the very favorable proportion of $h = \frac{a}{2}$, and also supposing the theoretically very favorable but impossible arrangement of an infinite number of panels, with all web-members placed at 45 degrees, the roof-truss would give $pa^2 \,.\, 1.625$, and the cantilever of uniform depth would yield $pa^2 \,.\, 1.666$. But for $h = \frac{a}{3}$ the figures become already $2\frac{1}{12}$ and 2 respectively, and for $h = \frac{a}{4}$ the cantilever built as a roof-truss has the coefficient 2.56, whereas the regular truss has only 2.33.

For a concentrated load, P, at the end of the triangular cantilever with $h = \frac{a}{2}$ there will be $Sl = 5Pa$, but for the cantilever with uniform depth it will be only $4Pa$. For $h = \frac{a}{3}$ the numbers are 6.66 and $5Pa$, and for $h = \frac{}{4}$ they are 8.5 and 6.12.

Hence it is clear that the triangular cantilevers cannot be selected for the sake of economy.

In practice the case stands still worse for the cantilever

with varying depth. For the webs of girders with parallel chords can be built to agree much better with the scientific minimum than happens with girders of varying depths.

Hence cantilevers with varying depth, in order to make good their unfavorable conditions, must be built of enormous depths; their end-posts become very long, the web-members act under angles still less favorable, and the theoretical advantage of small web-strains is sacrificed by the necessity of making long struts sufficiently rigid.

If the cantilevers were not built as full triangles, but in the shape of *ABCD* of Fig. 13, we should find

$$H = \frac{pa \, . \, x^2}{2\,[h_0 \, . \, a + x\,(h_1 - h_0)]},$$

and the shearing force absorbed by the chords would be

$$\frac{px^2}{2\left(\frac{h_0}{h_1}(a+b) + x\right)},$$

which is less than one half of the total shearing strain, px.

In order to show how much, on the average, the web-members will be relieved, let an example be calculated.

Let h_1 be equal to $7h_0$, $a = 14.5h_0$.

If the average of the shearing forces contained in the chords for $x = 0 \frac{a}{4}, \frac{a}{2}, \frac{3a}{4}$, and for $x = a$ is calculated, there will be found 0.39 of the average total shearing force $\frac{pa}{2}$, so that 61 per cent remains in the web.

This reduction is obtained at the expense of an increase of strain-lengths of the chords from 100 of the regular truss to 150 of the cantilever with varying depth; which proves again

that whatever there may be gained theoretically in the web is not only lost again partly in the practical execution of a web of varying depth, but that it is more than lost already in the chords.

§ 18. Best Arrangement of Web-Members of a Cantilever with Varying Depth.

Since cantilevers of varying depths cannot always be avoided, we shall now exiamne how the evil effect of varying depths can be reduced by a proper arrangement of web-members.

It is assumed that the length of the longest compression-member is decided upon. We place the posts in vertical positions. The question remains, how the diagonal ties must be arranged so as to lead to a minimum of strain-lengths of the web of a cantilever with varying depth.

In order to simplify the problem an auxiliary problem is introduced. B and F (Fig. 14) are to be considered as given points. In the line AB, at the distance z from EF, there acts a force P.

What must be $EF = h$ that the sum of strain-lengths of BE, BF, and of EF may become a minimum? We find easily

$$Sl = \frac{P}{h}\left[2z^2 + b^2 + (h-b)^2 + h(h-b)\right],$$

and a minimum is obtained for $h^2 = b^2 + z^2$, and the minimum is $P(4h - 3b)$.

Hence BE must be made equal to h, and the *line BF must bisect angle ABE.*

We return now to the cantilever $ABCD$ (Fig. 15). The sides BC and AD are produced until they meet in O. We

make $BK = AB = h$, and we draw the lines f parallel to h, and draw line $AK = e$.

The diagonal D parallel to AK gives point F. It bisects the angle ABE. The first post will be $FE = h_1$. In a similar manner ED is parallel to e, DC is the second post, and so on.

We shall prove that by this construction the minimum strain-length of the cantilever is obtained, supposing, as we have done, that all posts shall be in a perpendicular position.

The force P_1 is supposed to act in line AB.

We draw the imaginary line $BG = h_1$, and we find that the force in $BE = d$ must be $P_1 . \frac{d}{h_1}$, and the force in D must be $P_1 . \frac{D}{h_1}$.

Further, the force in EF is $P_1 . \frac{q}{h_1}$, less the part absorbed by the top-chord FD. This rest is $P_1 . \frac{h}{h_1}$, which is the pressure in post h_1.

The strain-length of the sides of triangle BEF is

$$Sl = P_1 . \left(\frac{D^2 + d^2}{h_1} + h\right).$$

We can transform Sl so that it contains only the unknown quantity h_1. There is

$$D^2 = h_1^2 + d^2 - 2p . h_1,$$
$$d = b . \frac{h_1 - h}{h}; \quad q - h = \frac{n . d}{b}; \quad p = m . \frac{d}{b}.$$

But what we wish to be a minimum is not Sl, but $\frac{Sl}{d}$, so

that after the introduction of the various new quantities we find that the expression

$$2b\left(\frac{1}{h} - \frac{1}{h_1}\right) + \frac{2h^2}{b(h_1 - h)} + \frac{n - m}{b}$$

must be made a minimum.

This minimum appears if $h_1 = d = h \cdot \frac{b}{b-h}$, or $D = e \cdot \frac{b}{b-h}$. The corresponding value of Sl is

$$Sl \text{ min.} = P_1 \cdot \frac{b}{b-h}\left(\frac{e^2 + h^2}{h} + f\right) = P_1 \cdot \frac{b}{b-h} \cdot \varphi.$$

In the following panel we shall have a new load P, consisting of $P_1 \cdot \frac{b-h}{b}$ plus the panel-load P_2 acting in point E, so that there is

$$Sl = \left[P_1 \cdot \left(\frac{b}{b-h}\right)^2 + P_2 \cdot \left(\frac{b}{b-h}\right)\right] \cdot \varphi,$$

and so on.

We saw that the part of P_1 which acts in the post h_1 is

$$P_1 \cdot \frac{h}{h_1} = P_1 \cdot \frac{b-h}{b} = P_1 \cdot \frac{OK}{OB}.$$

If this load P_1 were the only load of the cantilever, the posts h_1, h_2, h_3 would receive less and less pressure; the pressures would be

$$P_1 \cdot \frac{b-h}{b}; \qquad P_1 \left(\frac{b-h}{b}\right)^2; \qquad P_1 \cdot \left(\frac{b-h}{b}\right)^3, \text{ etc.}$$

It is also noticed that the panel-lengths increase in a geometrical proportion. The cantilever would assume the very objectionable form of being of unequal panels.

The strain in FD is $P_1 . \frac{c}{h_1}$, and if the chords OD and OC are inclined at equal angles, c must be $= d = h_1$, and the chord-strain in FD equal to P_1 itself.

In case the panels become too long additional systems may be introduced, which plan we shall illustrate by an example.

Fig. 16, Plate III., represents a cantilever of varying depth. The bottom-chord is horizontal, $\frac{b}{b-h} = 2$, and $b = 2h$.

By introduction of additional systems all panels of this cantilever become equally long, the diagonals will run at 45 degrees, and all panel-loads will be equally great.

Assuming separate anchorages for 1, 2, 3, 4, the least value of combined strain-lengths is obtained as follows:

For the bottom-chords,	35 Pd;	} chords,	$52.5P.d.$
" " top-chords,	17.5 "		
" " posts,	24.5 "	} web,	$66.5P.d.$
" " diagonals,	42.0 "		

Hence Sl is $2.43pa^2$.

This good result is to be attributed to the fourfold system of diagonals, which are all placed at the favorable angle of 45 degrees.

If the height of a seven-panel Poyet cantilever were also made $\frac{4a}{7}$, the value $Sl = 2.31pa^2$ would be found.

§ 19. Inclined Web-Posts of a Truss with Non-Parallel Chords.

In this section it is intended to find between non-parallel chords AB and CD, and also between the given panel-points A and B, the most favorable position of the web-members AE and BE (Fig. 17, Plate IV.).

The question of economical position of web-members has often been treated. But generally the error was fallen into of considering the two web-members alone, instead of considering *all the changes which take place in a panel.*

Reference is had to Fig. 17, *a*, Plate IV.

The shearing force in panel AB is the algebraic sum of all loads, including the vertical reaction to the right side of B, and also including the panel-load acting in B, no other forces but weights acting vertically being supposed. The moment of exterior forces for BC being M_1, and the moment of exterior forces for AD being M_2, the following equation is known to exist:

$$M_2 = M_1 + S \,.\, l,$$

where S is the shearing force.

Of this shearing force only a part is transferred by the diagonals unless the chords are parallel, and this case is here excluded.

The parts of S which are absorbed by the chords in points cut by an imaginary vertical line infinitely near to the right side of B are

$$\frac{M_1}{h_1} \cdot \frac{h_2 - h_1}{l};$$

and since
$$S = \frac{M_2 - M_1}{l}, \quad \ldots \ldots \ldots \quad (24)$$

there remains a force

$$P_1 = \frac{M_2 - M_1}{l} - \frac{M_1}{h_1} \cdot \frac{h_2 - h_1}{l} = \frac{M_2 \cdot h_1 - M_1 \cdot h_2}{h_1 \cdot l}. \quad (25)$$

In a similar manner, the part of the shearing force S which is absorbed by the chords in points A and D (after the force EA has acted) is

$$P_2 = \frac{M_2 - M_1}{l} - \frac{M_2}{h_2} \cdot \frac{h_2 - h_1}{l} = \frac{M_2 \cdot h_1 - M_1 \cdot h_2}{l \cdot h_2}. \quad . \quad (26)$$

Also, this force can be used to determine the changes taking place in the panel; but if we start with P_2, we must commence at point A instead of point B.

If the chords were parallel, h_2 would equal h_1, and P_1, P_2, and S would be equal to each other, equal to $\frac{M_2 - M_1}{l}$.

The forces P_1 and P_2, in case of a truss with non-parallel chords, fulfil the office of the whole shearing force in a truss of parallel chords. Especially they cause the changes in the chord-strains. There is

$$\frac{M_1}{h_1} + \frac{P_1 \cdot l}{h_2} = \frac{M_2}{h_2} \text{ and } \frac{M_2}{h_2} - \frac{P_2 \cdot l}{h_1} = \frac{M_1}{h_1}, \text{ or } P_1 \cdot h_1 = P_2 \cdot h_2. \quad (27)$$

This equation also results from (25) and (26).

Fig. 17 is a representation of a cantilever-truss, in which P causes pressure in AB and tension in EB. But the rules of this section hold good also for single-span trusses. It is only necessary to assume a negative force P_1, because in case of an ordinary truss P_1 and P_2 would have the sign of the reaction, which is negative if the panel-load has the sign plus.

We draw the auxiliary line EK parallel to $BC = h_1$, and

we make BG parallel to CD. The sides BK, BE, and EK of triangle BEK are parallel to the directions of the bottom-chord strains, of the diagonal tension, and of the force P_1. Hence if EK by its length represents the force P_1, BK and BE by their lengths represent the other two forces. If the same principle is applied to the force T of the top-chord,

which force equals $P_1 . \frac{l}{h_2} . \frac{d}{l} = P_1 . \frac{d}{h_2}$, and to the force V in

the diagonal AE, the sides of triangle BEG will be found to represent the forces x, T, and V.

Also, P_2 can be shown graphically; for if GK_1 is made parallel to AB, the line EK_1 must be this force. The equations (27) admit of still another interpretation. There is

$$P_1 . \frac{l}{h_2} = \frac{P_2 . l}{h_1} = \frac{M_2}{h_2} - \frac{M_1}{h_1} = P_h = T . \frac{l}{d}.$$

This is the horizontal shearing force contained in the two diagonals BE and EA.

The force P_h, resulting directly from the calculation of strains of structures such as here considered, is easily utilized towards a graphical representation of the forces in the panel $ABCD$. The horizontal projection of BG is this force.*

* The above constructions have some relations to the one referred to in § 12. They rest upon the principle of the triangle of forces. The Dutch philosopher Simon Stevin of Bruges gave this law in 1586, not merely, as has recently been erroneously stated, for rectangular triangles—such as done by Michael Varro of Geneva in 1584, and by Galileo in 1592—but for any triangle. Stevin's books were repeatedly edited and translated into German and into French; among others by Albert Girard (published in Leyden in 1634). Stevin was also the first mathematician who represented forces by lines, and he applied his principle to pulleys and to funicular polygons (his "spartostatics," or statics of cordage). The principle of the triangle of forces is repeatedly and distinctly

The strain-length of the changes in panel $ABCD$ is

$$P_1 \cdot \frac{a \cdot BK + x^2 + T \cdot ED + y \cdot EG}{h} = Sl,$$

and must be made a minimum.

The unknown lengths can be expressed by $OE = \rho$. There is

$$BK = \rho \cdot \frac{a}{d} - b; \qquad x^2 = b^2 + \rho^2 - 2p \cdot \rho,$$

where $p = OL$ is the projection of b upon the top-chord.

$$EG = y \cdot \frac{h_1}{h_2}; \qquad y^2 = (a+b)^2 + \rho^2 - 2\rho \cdot p_1,$$

where p_1 is the projection OH of $(a+b)$ upon the top-chord.

$$T = \rho \cdot \frac{a}{a+b}; \quad \text{hence } T \cdot ED = \rho \frac{a}{a+b}(c + d - \rho).$$

Finally there is $h = \rho \frac{h_1}{c}$.

These values introduced in Sl give

$$\frac{2b^2c}{h_1 \cdot \rho} + \frac{c}{h}\left(\frac{a^2+d^2}{d} - 4p\right) + \rho \cdot \frac{2c}{h^2};$$

---

stated and applied by him, and the figures illustrating his spartostatics also contain the parallelogram of forces. Since graphical statics mainly rest on the properties of funicular polygons, it seems well to remember the claims of Stevin, which are older than those of Varignon. Varignon's and Newton's books appeared in the same year, 1687, or over a hundred years later than the "Beghinselen der Weegconst," and Varignon in his book defends Stevin against doubts raised by Borelli.

which for $p = b.\ \sqrt{\frac{h_2}{h_1}} = \sqrt{b(a+b)}$ becomes a minimum. (28)

$$Sl\ \text{min.} = \left(\frac{4b\,.\,c}{\sqrt{h_1\,.\,h_2}} + \frac{c}{h_1}\left(\frac{a^2+d^2}{d} - 4p\right)\right).\ P_1.$$

We also have $$h = \frac{a}{d}\sqrt{h_1\,.\,h_2}$$

and $$ED = \frac{d\,.\,h_2 - a\sqrt{h_1\,.\,h_2}}{h_2 - h_1}.$$

If the chords are parallel (but not necessarily horizontal), the strain-length (see Fig. 18, Plate III.) becomes

$$\frac{x^2 + y^2 + ua + (a-u)a}{h}\,.\,P.$$

This expression becomes a minimum if $x^2 + y^2$ is a minimum, and this happens if $DE = \frac{a}{2} \pm$ projection of h upon the chords.

This result can also be drawn from the above expression of

$$DE = c + d - p.$$

Especially if the chords are both horizontal,

$$ED = \frac{a}{2} \quad \text{and} \quad Sl = P\left(2h + \frac{3a^2}{2h}\right). \quad . \quad . \quad (29)$$

If the chords are not horizontal but only parallel, a line drawn at right angles through the middle of AB meets the point E and makes x equal to y. The strain-length in this case is

$$Sl\ \text{min.} = P\left(2h + \frac{3a^2}{2h} - \frac{2.\,\varepsilon^2}{h}\right), \quad . \quad . \quad . \quad . \quad (30)$$

where ε is the projection of h upon the chord. The web of such a truss is light, but the chords are heavy.

If the chords were supposed to be horizontal, and u (Fig. 18) were made $= 0$, the strain-length would be

$$P.\left(2h + \frac{2a^2}{h}\right).$$

The absolute minimum for horizontal chords and a specified panel-length appears if

$$h = \frac{a}{2}\sqrt{3},$$

or if $$AE = BE = a.$$

This is the case of the triangular truss with equilateral triangles.

In this case the strain-length of the changes within a panel is found $3.464P.a$, where P now is the whole shearing force.

If $u = 0$ and $a = h$, the minimum obtained is $4aP$.

This is the case of the Pratt truss with diagonals at 45 degrees.

The shearing force P causes only a part of the chord-strains of the panel considered. Indeed if the whole strain-length of the truss has to be found, the consideration of the diagonals alone, or of the changes per panel alone, is insufficient.

We have seen in §9 that the coefficients of strain-lengths of a regular Whipple truss (13) are better than those of the triangular truss (10).

Even if the triangular truss were built with the same height, $h = 2d$, it would be found to have a greater strain-length than the regular Whipple truss.

Only if the panel-number reaches 18, the coefficients of both types become equal. For smaller panel-numbers the Whipple has the theoretical advantage added to practical superiority.

Thus in all our investigations it has been found that there is no advantage in complexity of design, and especially there was none in trusses of varying depth, with the only exception of the modifications treated in § 11. And even there the advantage was not much more than the reduction of chord-strains in the centre of a bridge by adopting still greater depths and greater panel-lengths than usual with regular Whipple trusses.

§ 20. Anchorage of Cantilevers.

The anchorage-ties of a cantilever may be arranged with or without horizontal or inclined back-struts.

The locality may be of such a nature as to admit of the arrangement of a *horizontal* tension-member by which the top of the end-post of a cantilever is to be anchored to rock or to masonry. In most instances, however, this arrangement, more or less economical according to the length of the retaining-tie, is not possible. The retaining-tie assumes an inclined position towards a lower point, where the connection is made with the thrust-member and vertical anchors, or with a block of masonry, or with available natural anchorage. These inclined back-ties will create pressure upon the end-posts or tower of the cantilever.

The additional material in the end-post, in the back-ties, and in the vertical anchorage-bolts, and also of the masonry, must be so arranged as to lead to a minimum of cost.

Some of the forms of anchorage will now be examined.

(*a*) Cantilevers without back-struts (Fig. 19, Pl. IV.).

The lower member of the cantilever abuts at A. The end-post is represented by $AC = h$. The maximum moment of flexure of the cantilever is denoted by M, and BC is the back-tie.

At present we only calculate the strain-length as far as B. We find

$$\left.\begin{array}{l} Sl = M \cdot \left(\dfrac{h}{z} + \dfrac{z^2 + h^2}{h \cdot z}\right) = M \cdot \dfrac{z^2 + 2h^2}{h . z}; \\ \text{which for } z = h \cdot \sqrt{2} \text{ leads to the minimum} \\ Sl = 2\sqrt{2} \cdot M. \end{array}\right\} \quad . \quad . \quad (31)$$

It is to be noticed that Sl minimum is independent of the height h; it only depends on M, an observation good for all variations of anchorages of cantilevers.

The amount of masonry at B is no longer independent of z or h. The horizontal pull at B is $\frac{M}{h}$, and it must be resisted by friction of the block of masonry on the foundation. If the coefficient of friction is 0.7, the effective masonry must weigh $\frac{10}{7}\frac{M}{h}$ to furnish a bare equilibrium.

The force $\frac{M}{z}$ represents the counterweight needed at B. If z is made $1.42h$, as per formula (31), it will be sufficient to provide against sliding, for $\frac{10}{7}\frac{M}{h}$ already gives a factor of safety of 2 as regards lifting.

If the proportion of $\frac{h}{z}$ is retained, the strain-length will remain the same, however great h is made. But by increasing h, z also will increase, and the masonry will become less and less the greater h is made.

On the other hand, the back-ties must be prevented from sagging, and must be stiffened laterally; so that an increase of h means increased cost in this respect.

If there exists natural anchorage, vertical bolts may have to be used. If they are of the length $\frac{h}{2}$, the minimum of strain-length will be $3.16M$ for $z = h \,.\, \sqrt{2.5}$.

(*b*) To Fig. 19 a horizontal strut is added.

$$\text{There is} \qquad Sl \text{ min.} = 4M, \qquad \text{for} \qquad z = h. \quad . \quad . \quad . \quad (33)$$

If also vertical bolts of the length h were added, the minimum strain-length would be $4.48M$ for $z = 1.118h$.

For cantilever anchorage with back-struts the masonry will be reduced. The difference of effective weight of masonry for the factor of safety of 1 is $\frac{3}{7}\frac{M}{h}$, against which there is an increase of strain-length of

$$(4.00 - 2.83)M = 1.17M.$$

If the weight of one M of strain-length costs c, while $\frac{M}{1 \text{ foot}}$ of masonry costs c', the question arises whether $1.17 \,.\, c$ or $\frac{3c'}{7h}$ is the greater of the two.

Apparently, beyond $h = 3.67 \,.\, \frac{c'}{c}$ masonry becomes more economical, unless the cost of foundation also enters into the calculation.

(*c*) In many localities an arrangement like that of Fig. 20 will be necessary.

In this case the strain-length of AC, CD, DE, and BD is

$$Sl = 2M \cdot \left(\frac{z^2 + h^2 + h_1^2}{zh}\right);$$

which for $z = \sqrt{h(h + h_1)}$ becomes a minimum,

$$\frac{4M}{h} \sqrt{h(h + h_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

Especially for $h = h_1$ there is $z = h \cdot \sqrt{2}$, and

$$Sl \text{ min.} = 4\sqrt{2}M.$$

This is double the value (31).

(d) The back-strut AD is inclined (Fig. 21, Pl. IV.).

$$Sl = \frac{2M}{z(h_1 + h_2)}(z^2 + h_2^2 + h_1 \cdot h_2 + h_1^2).$$

For $z = \sqrt{h_1^2 + h^2 + h_1 \cdot h_2}$ the minimum is

$$Sl = 4\frac{M \cdot z}{h_1 + h_2}.$$

For $h_1 = h_2$ there is $z = h_1 \cdot \sqrt{3}$, and Sl min. $= 3.46M$. Or

$$AD = AC = 2h_1 = h.$$

The maximum moment M for cantilevers of extraordinary spans, of which the permanent weight is very great near the main-posts, is at the best $\frac{3}{8}pa^2$. If the middle span were one half of the whole span, the moment M would always be less

than $\frac{3}{8}pa^2$. Even if $\frac{3}{8}pa^2$ were taken as a basis of calculating the strain-lengths of the anchorages of cantilever-bridges, the coefficients of these strain-lengths would still be very considerable, as compared with those for Whipple bridges.

We find for arrangements:

	a	b	c	d	
$Sl = pa^2 \times$	2.12	3.00	4.24	2.60	. . . . (35)

This result confirms the remark at the end of § 16.

§ 21. Utilization of Anchorage-Material for Short Outside Spans of a Cantilever-Bridge.

The anchorage of a cantilever with back-struts may be utilized to build a small outside span. CF of Fig. 20 is one of many stays carrying the back-strut DE, and also carrying the movable load and the floor of the short span.

The anchorage must be proportioned for the main span being loaded, whilst the short span DE is empty.

We suppose $DE = h\sqrt{2}$, such as found in § 20, c. The permanent load of the floor, etc., of the short span causes a small reduction of strain-length of the back-stays and of the strut, and a little more material in EC results in a reduction

$$- 1.9 p_0 h^2.$$

The diagonal stays of the short span are strained by p_0 as well as by the movable load, p_1, and their value of strain-length is

$$(p_0 + p_1)\int_0^{h\sqrt{2}} \frac{h^2 + x^2}{h} \cdot dx = \frac{5}{3}\sqrt{2}h^2 \cdot (p_0 + p_1).$$

The movable load adds $\frac{p_1 . h\sqrt{2}}{2}$ to the pressure of the main-post AC, giving

$$p_1 \frac{h2}{\sqrt{2}}.$$

The result is an increase of strain-length equal to

$$3.07 p_1 h^2 + 04.6 p_0 h^2.$$

Approximately it may be considered that the whole movable load $p_1 . h\sqrt{2}$ of the short span has travelled through a little more than the height $2h$, and that this load and the load $p_0 h\sqrt{2}$ has still to pass through the height h_1.

Nevertheless there may be an advantage in using the short span. This is so because it may be difficult, or at least costly, to make the masonry for the trestle-posts, and because a viaduct would require much longitudinal and lateral bracing.

For a fixed height h of the cantilever there exists a length z of the short span, which makes the expense of iron-work and masonry of the anchorage a minimum. But the height may also be varied. This variation does not affect the minimum value of the strain-length, but it affects the masonry. Only if the greatest admissible height h has been reached will it remain to examine for what z anchorage metal-work plus additional material for the short span plus masonry, if any, becomes a total minimum.

To make the anchorage longer still, under the impression of gaining counterweight, would be unwarrantable, because each additional ton of iron would cost more than four times the equivalent counterpoise of stone.

Finally, the reduction of weight is no good reason for making the *anchorage* of steel. Only if the price of well-executed

steel work were so moderate that in consideration of higher strains, specified or admitted, the unit of strain-length would become cheaper than the unit executed in iron, would it be economical to build the anchorage of steel.

Between the piers matters are wholly different. There the weight of the structure becomes very important for long spans, and material which admits of higher strains per square unit of section, though more expensive per unit of strain-length, may still be preferable.

§ 22. The Strains Produced by the Weight of the Carrying-Frame of a Truss. The Limiting Spans.

If the strain-length of a certain type of truss is multiplied by a certain constant coefficient, the weight of the skeleton-truss is found.

The strain-lengths as given in the preceding paragraphs do not contain the material providing against crippling of long struts, nor for increased web-strains under progressing movable loads, nor for connecting plates, rivets, bolts, pins, etc. All this material must be included in that coefficient of experience.

The weight $c_{,}$ for 12 cubic inches of theoretical volume of a carefully designed, large pin-jointed Whipple truss may be assumed to be $4\frac{3}{4}$ pounds. (1000 cubic centimetres = 1 cub. decm. = 11 kilo.)

If the strain-length is $Sl = c \cdot p' \cdot a^2$, where c is one of the co-efficients calculated in the previous paragraphs, and where p' is the weight per foot of the length of the carrying-frame itself, we wish to find the strain per square inch which is caused by the weight p'.

The strain-length per lineal foot of the span is

$$\frac{c}{4}(2ap').$$

This quantity divided by the uniform strain x gives the volume of a foot of length of the carrying-frame. And this volume $\frac{c}{4 \cdot x} \cdot 2ap'$ multiplied by the coefficient $c_{,}$ gives again p'; or there is

$$p' = \frac{c \cdot c_{,}}{4 \cdot x} \cdot 2ap',$$

or

$$x = \frac{cc_{,}}{4} \cdot 2a = \frac{c \cdot c'}{4} \times \text{Span}; \quad . \quad . \quad . \quad . \quad . \quad (36)$$

or, *The strain caused by the weight of the carrying-frame itself increases as the span; it increases in the same ratio as the coefficient of strain-length, and as the coefficient of design* ($c_{,}$).

If s is the specified maximum strain of the structure, the difference $(s - x)$ is the strain available for the strain-length of the movable load.

As soon as x has become as great as s itself there is nothing left for the movable load, and the *limiting span* for the *specified strain* s is reached. Or the

$$\text{Limiting span of } 2a = L = \frac{4 \cdot s}{c \cdot c_{,}}. \quad . \quad . \quad . \quad . \quad (37)$$

This is certainly a very simple formula.

The limiting span increases in the same ratio as the specified maximum strain, and

The limiting span decreases in the same geometrical ratio as the product of the coefficients of strain-length and of design increases.

For $c_{,} = 4\frac{8}{4}$, and for a Whipple truss of ten panels, for which c was found to be 5.96, the limiting span must be

$$L = \frac{4 \cdot s}{5.96 \cdot 4\frac{8}{4}};$$

and if the specified strain s is 5 tons $= 11,200$ pounds, there must be $L = 1580$ feet.

The coefficient $c_{,} = 4\frac{8}{9}$, which in general is variable with the character of detail design, was found for a 400-foot, 18-panel Whipple truss, 50 feet deep, arranged with eye-bars for tensional members, and with wrought-iron hollow segment columns as vertical posts.

The strain-length of such a span (with triple system of diagonals) would be $7.95pa^2$, and the span would not contain any secondary bearing or latticing. Its girders would weigh just about one third of the whole load carried on the panel-points, or the limiting span would be $\frac{4 \cdot 11200}{4\frac{8}{9} \cdot 7.95} = 1187$ feet, which is nearly three times the span of 400 feet.

There is another method of expressing the limited span of a bridge, which—so far as we know—was first given by Privy Councillor Schwedler of Berlin, in the year 1863.

If K represents the weight per lineal foot of the movable load plus that of the wind-bracing and of the floor, and if w is the weight of the skeleton-frames per lineal foot, the equation

$$w = \frac{2a}{y} \cdot (w + K)$$

obtains, where y is a certain unknown constant coefficient. Or we have this new formula:

$$w = \frac{2aK}{y\left(1 - \frac{2a}{y}\right)} = \frac{K}{\frac{y}{2a} - 1} \quad . \quad . \quad . \quad . \quad (38)$$

If $2a$ becomes equal to y, w will be infinitely great.

Therefore y is the limiting span L; and if w has been ascer-

tained from one single bridge of a certain type, L can be calculated, namely,

$$L = 2a\left(1 + \frac{K}{w}\right) = \frac{4s}{c.c_{\prime}}; \quad . \quad . \quad . \quad . \quad (39)$$

$$w = K.\frac{1}{\dfrac{2s}{a.c.c_{\prime}} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It will be noticed that if w and $(2a)$ are considered as the co-ordinates of a curve, this curve will be an hyperbola.

If the length of span, its number of panels, depth and system are assumed, the coefficient of strain-length c is known. If, then, the style of details is fixed upon, also $c_{\prime}$ is known, and w can be calculated directly from formula (39) for any given value of K.

For this reason the weight of the floor and of the wind-bracing of a bridge of considerable span should be first calculated. These weights determine K. Thereupon the weights calculated by formula (39) will be found quite sufficient to make the exact calculation, which need not be repeated.

It is also seen how formula (40) may be used to calculate from the weight w of a certain span the value of $c_{\prime}$, or the coefficient of detail-design.

§ 23. A Whole Cantilever-Bridge.

The principles of the last section will now be applied to a whole cantilever-bridge.

But before doing this we shall look at the value of a whole cantilever-bridge in a less intricate manner.

Let the middle span be $2b$ long, each cantilever-arm to have the length $(a-b)$.

In order to represent an example very favorable for the pure cantilever plan, we assume for the middle span a depth of one tenth, but for the cantilevers themselves we assume the much greater depth of one third of $(a-b)$. Thereupon the strain-lengths are found as follows:

For the middle span, Whipple truss.......... $9.4\, b^2 . p$.
For the cantilevers, arising from the middle span resting on the cantilevers............ $12.0 . (a-b)b . p$.
For the cantilevers, arising from their own loads.................................. $6.3(a-b)^2 . p$.
For the anchorages (type *c*, Fig. 20), the minimum value for uniform load.............. $5.6(a^2-b^2)p$.

These values added together give

$$Sl = pa^2\left[11.9 - 0.6\frac{b}{a} - 1.9\left(\frac{b}{a}\right)^2\right].$$

This function has no minimum.

The *maximum strain-length is found for the pure cantilever.*

The greater the middle span is made the better the result. And if the cantilevers are suppressed altogether, also the anchorages disappear with masonry and with back-struts, and we obtain the *best and most economical design*, and this is the *pure Whipple truss.*

In the foregoing calculation we have not yet considered the *variable* quantity p, nor shall we do this in the following example:

We give to the middle span a greater depth; for instance, the very favorable depth of one seventh of the span. Further, we shall use the very favorable anchorage with back-struts inclined at the most favorable angle. We shall also consider that in case of very great spans the maximum moment of flexure of a cantilever is only three fourths of the

moment obtained for uniform load. We make, finally, the greatest depth of the cantilever equal to $\frac{a-b}{2}$.

Now we obtain:

For the middle span (depth 1 : 7)............. $7.32pb^2$.
For the cantilevers, from load of middle span.. $10.00p(a-b)b$.
For the cantilevers, from their own loads.... $6.00p(a-b)^2$.
For the anchorages, $\frac{3}{4}\,.\,3.46(a^2-b^2)\,.\,p$ .... $2.60p(a^2-b^2)$.

Hence

$$Sl = pa^2 \,.\, \left[8.6 - 2\frac{b}{a} + 0.72\left(\frac{b}{a}\right)^2\right].$$

Also, this expression has no mathematical minimum between the values $b = a$ and $b = o$.

But the differential quotient $\frac{dSl}{db}$ is negative, and Sl becomes less and less the greater $2b$ is made; and if we make the middle span $2a$ long we obtain again the best result, namely, the Whipple truss without any cantilevers.

The pure cantilever gives $8.6pa^2$, and the pure Whipple truss gives only $7.32pa^2$.

These are the results if constant values p are supposed, which supposition is correct enough for small and for moderate spans.

Hence we have at once the rule:

For small and for moderate spans the cantilever-bridge is inferior to the Whipple truss, unless a Whipple truss could not be built because of the difficulty of erection.

The advantage of the system of the cantilever-truss must be sought for in other qualities than the *total* strain-length.

The material *between* the piers which causes the strains from permanent loads is small, for it depends, not on the whole strain-length, but only on the strain-lengths between the piers.

This is a matter of great importance for great spans.

The strain-length between the piers in above example is

$$a^2p\left[b - 2\frac{b}{a} + 3.32\left(\frac{b}{a}\right)^2\right],$$

and this becomes a minimum for $b = \frac{a}{3}$, which minimum is $5.7pa^2$, against the total strain-length of $8.0pa^2$. Consequently the permanent load $(p - K)$ must become much smaller than would happen for a Whipple truss. Therefore a cantilever-bridge to be economical must be designed in such a manner that the carrying-frame be very light, and this is achieved by using the smallest practicable number of panels.

Now we are prepared to enter into a more complete investigation of the weights of cantilever-bridges.

We denote with

K, the movable load, the wind-bracing, and the floor, per unit of length of the bridge (per lineal foot).

$c_{\prime}$, the coefficient of design of the carrying-frame.

$c = 7.32$, the coefficient of strain-length of a Whipple truss of fourteen panels, with $h = \frac{2a}{7}$.

w, the weight per lineal foot of the carrying-frames of the middle span.

K_1, the weight per lineal foot of the cantilevers, as arising from the weight of the middle span.

K_2, the weight of the cantilevers, as arising from their own weights.

s, the specified strain per square unit.

We have

$$w = \frac{K.3.66.b.c}{s - 3.66.b.c_{\prime}}; \qquad 2bw = \frac{7.32b^2.c_{\prime}.K}{s - 3.66.b.c_{\prime}}.$$

The load carried at each end of the cantilevers is

$$b\,.(K+w)=\frac{b\,.\,K\,.\,s}{s-3.66\,.\,b\,.\,c_{\prime}}.$$

For the cantilevers we assume the depth $\frac{a-b}{3}$, with $c=6$ for concentrated loads at the ends [see table (21)].

The weight K_1 of the cantilevers will be

$$K_1=\frac{6b\,.\,c_{\prime}\,.\,K}{s-3.66b\,.\,c_{\prime}}.$$

The total load per lineal foot of a cantilever is

$$(K+K_1+K_2) \qquad \text{and} \qquad K+K_1=K\,.\frac{s+2.34b\,.\,c_{\prime}}{s-3.66b\,.\,c_{\prime}}.$$

The coefficient c, belonging to the strain-length of the cantilever under uniform load, may be assumed (for $h=\frac{a-b}{3}$ see table (23) of §16) as 6.3 for both cantilevers. Hence, utilizing formula (36), there is obtained

$$2(a-b)K_2=6.3\,\frac{c_{\prime}}{s}\left(K+\frac{6b\,.\,c_{\prime}\,.\,K}{s-3.66\,.\,b\,.\,c_{\prime}}+K_2\right)(a-b)^2,$$

or $$K_2=\frac{3.15\,.\,c_{\prime}}{s-3.15\,c_{\prime}(a-b)}\,.\,K\,.\frac{s+2.34\,b\,.\,c_{\prime}}{s-3.66\,b\,.\,c_{\prime}}(a-b).$$

The weight of the whole cantilever-bridge between the piers, therefore, will be

$$2b\,.\,w+2(a-b)(K_1+K_2)=\frac{K\,.\,a}{\frac{s}{ac_{\prime}}-3.66\frac{b}{a}}\,.$$

$$\left(12\,\frac{b}{a}-4.68\left(\frac{b}{a}\right)^2+6.3\left(1-\frac{b}{a}\right)^2.\frac{1+\frac{2.34}{s:ac_{\prime}}\cdot\frac{b}{a}}{1-\frac{3.15}{s:ac_{\prime}}\left(1-\frac{b}{a}\right)}\right).$$

This expression becomes infinitely great if

(*a*) $s = 3.66b \, . \, c_{\prime}$, or if $b = \dfrac{s}{3.66 \, . \, c_{\prime}}$,

or if the middle span is equal or longer than its limiting span;

(*b*) $s = 3.15(a - b)c_{\prime}$, or $a - b = \dfrac{s}{3.15c_{\prime}}$,

or if the cantilever reaches its limiting span.

The anchorage moment will be found

$$M = (a - b)b(w + K) + \left(\frac{a - b}{2}\right)^2 (K + K_1 + K_2)$$

$$= (a - b)\left[\frac{b \, . \, K}{s - 3.66b \, . \, c_{\prime}} + \frac{K_2}{6.3 \, . \, c_{\prime}}\right] . \, s,$$

where K_1 and K_2 are considered as uniformly distributed, so that in reality M will be smaller.

The weight of two anchorages with inclined end-struts, but exclusively of their vertical and horizontal bracing, will be

$$\frac{6.92 \, . \, M : a}{s : ac_{\prime}} = 6.92 \, . \, c_{\prime} \, . \, (a - b) \, . \left[\frac{b \, . \, K}{s - 3.66b \, . \, c_{\prime}} + \frac{K_2}{6.3 \, . \, c_{\prime}}\right].$$

The anchorage weight thus found will be approximately correct; for the two errors committed will almost neutralize each other.

We shall now use the above formulæ to determine the weights of a cantilever-bridge for various proportions of $\dfrac{b}{a}$.

We assume $2a = 1700$ feet, $s = 6$ tons $= 13440$, $c_{\prime} = 5$ pounds, $\dfrac{s}{a \, . \, c_{\prime}} = 3.16$, $\dfrac{s}{a} = 15.81$, and obtain for

$\frac{b}{a} =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$w = K \times$	0.131	0.301	0.532	0.862	1.374	2.273	4.270	12.51	∞
$K_1 = K \times$	0.214	0.493	0.872	1.413	2.253	3.726	7.00	20.55	∞
$K_2 = K \times$	10.59	5.89	4.300	3.61	3.24	3.13	3.40	36.8	∞

Total Weights.

Middle span:

$Ka \times$	0.026	0.120	0.319	0.690	1.374	2.527	5.980	20.00	∞

Two cantilevers, $2\dfrac{(a-b)}{a}\dfrac{K_1+K_2}{K} \cdot Ka$:

$Ka \times$	19.48	10.21	7.24	6.02	5.40	5.49	6.24	22.02	∞

Together the whole span between the piers:

$Ka \times$	19.51	10.33	7.56	6.71	6.76	7.92	12.22	42.92	∞

The anchorages,

$$2.188\left(1-\frac{b}{a}\right)\left[\frac{b}{a}\left(1+\frac{w}{K}\right)+\frac{1}{2}\left(1-\frac{b}{a}\right)\left(1+\frac{K_1}{K}+\frac{K_2}{K}\right)\right] \cdot Ka:$$

$Ka \times$	10.68	5.62	4.01	3.35	3.07	3.06	3.54	36.4	∞

The whole cantilever-span:

$Ka \times$	30.19	15.95	11.57	10.06	9.83	10.98	15.76	79.32	∞

We shall now assume $s = 7$ tons instead of 6, and again calculate all values:

For $\frac{b}{a} =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$w = K \times$	0.11	0.25	0.42	0.66	0.98	1.47	2.27	4.11	8.32	122
$K_1 = K \times$	0.18	0.41	0.70	1.08	1.63	2.41	3.72	6.31	13.60	0
$K_2 = K \times$	6.01	3.01	2.52	2.18	1.94	1.77	1.63	1.51	1.71	0
Middle span $= Ka \times$	0.02	0.10	0.25	0.53	0.98	1.76	3.18	6.58	14.98	244
Two cantilevers $= Ka \times$	11.14	5.46	4.50	3.91	3.57	3.34	3.22	3.13	3.06	0
Whole span between piers,	11.16	5.56	4.75	4.44	4.55	5.10	6.39	9.71	18.04	244
Anchorages $= Ka \times$	5.94	3.04	2.49	2.18	1.99	1.88	1.82	1.79	1.76	0
Whole bridge $= Ka \times$	17.1	8.6	7.24	6.62	6.54	6.98	8.21	11.50	19.8	244

In order also to see the influence of careful design, let c_t be assumed equal to 4.5, s remaining 7 tons.

There will be found for $\frac{b}{a} = \frac{1}{2}$:

$$\left.\begin{array}{ll}\text{Between the piers,} & Ka \times 3.56; \\ \text{For anchorages,} & Ka \times 1.59.\end{array}\right\} \text{ Together, } 5.15\, K.a.$$

If K were $3\frac{1}{2}$ tons, of which 2 tons are supposed to be movable load,

The weight of a 1700-foot span between the piers would be	10,591 tons.
To which add for floors and wind-bracing	2,550 "
And the anchorage	4,730 "
Total span for two tracks	17,871 tons.

The results of the above two tables of numerical calculations are represented in Figs. 29 and 30 of Plate II.

The calculation has shown that the best result is obtained if the middle span is nearly as long as the two cantilevers together, which result would agree with the common theory of the beam rigidly fixed at its ends.

If the average moment of a beam fixed at its ends shall become a minimum, supposing the cantilever-length to be x and the middle span to be $2(a - x)$, we have for the average moment of the middle span,

$$\frac{p}{3}(a-x)^2; \qquad Sl = \frac{2}{3}p(a-x)^3;$$

for the average moment of a cantilever as arising from the central span,

$$\frac{p}{2}(a-x)x; \qquad Sl = p(a-x)x^2;$$

for the average moment of a cantilever as arising from its own load,

$$\frac{p}{6}x^2; \qquad Sl = \frac{p}{3}x^3;$$

and the expression

$$\frac{(a-x)^3}{3} + (a-x)\frac{x^2}{2} + \frac{x^3}{6}$$

becomes a minimum or maximum for

$$x = a\left(\frac{3}{4} \pm \frac{1}{4}\right).$$

For $x = a$, pure cantilever, $Sl = p\frac{a^2}{3}$, the maximum;

For $x = \frac{a}{2}$, fixed beam, $Sl = p\frac{a^2}{8}$, the minimum.

The weights of cantilever-bridges increase but little if b remains in the neighborhood of $0.4a$.

If, for instance, the 1700-foot span were divided into thirty-four panels of 50 feet, the middle span would be 700 feet long, 100 feet deep, and would contain fourteen panels. Each cantilever would be 500 feet long, 100 feet deep at the end and about 250 feet deep at the towers.

Or else the bridge might be divided into twenty-seven panels of nearly 63 feet length, of which eleven would form the middle span, 693 feet long.

If b were made $= \frac{a}{2}$, the average maximum moment of horizontal flexure as caused by wind, and also the maximum moment at the pier as caused by wind, would be only three quarters of that of the pure cantilever.

It is easy enough to repeat the process of calculation for any other span or other arrangement of cantilevers and middle span, or for any other coefficient $c_{\prime}$.

Our formulæ may also be used to determine approximately for what span the carrying superstructure of a cantilever-bridge will be lighter than that of a Whipple truss.

To be exact it would be desirable to determine for each combination of s, $c_{/}$ and a the most favorable middle span, $2b$. But we shall here only assume that the middle span occupies one half of the opening; and assuming $s = 6$ tons and $c_{/} = 5$ pounds, we shall calculate the weights for the spans of 700, 600, and 500 feet.

We find:

	700	p. foot.	600	p. foot.	500	p. foot.
Central Whipple truss,	K.112	0.32	78	0.26	52	0.21
Two cantilevers,	K.312	0.89	216	0.72	140	0.56
Two anchorages with inclined struts,	K.203		112		75	
Total cantilever-bridge,	K.627		406		267	

On the other hand, Whipple trusses over the whole spans, depth one seventh of spans, would weigh K.641 414 258

According to these results, practice, which has produced single spans of 525 feet, has about reached the limit of economical length of single spans. But it is worthy of being noticed that up to 700 feet the cantilever-bridge is but little more favorable than the single span. Besides, cantilever-bridges require more masonry than single spans, but can be erected without false-works; and there being *more* panels and generally *lighter* pieces, and also there being no necessity for separate iron piers (according to the above suppositions), it will be necessary in each particular instance to enter into a special calculation.

The formulæ given will no doubt greatly facilitate this investigation.

§ 24. Examples of Cantilever-Bridges.

In designing, in the year 1879, the Magdalena bridge at Jirardot, in New Granada, some of the principles laid down in §§ 15 to 23 were utilized. The span is 328 feet (100 metres), and the cantilever plan was chosen not only on account of the difficulty, or even impossibility, of using false-works, but also because it was imperative to use less masonry than would have been required for a suspension-bridge.

Notwithstanding the great cost and difficulty of timely transportation on the upper part of the river, it was considered best to use iron piers above water, because this was the less of two evils.

Thus the author, who for many years had paid attention to this class of structures, designed the first cantilever-bridge with independent middle span. (See Fig. 31, Pl. V.). The depth of the cantilevers at the main piers is 41 feet, or one eighth of the whole span. The anchorage with back-struts is of the class *c*, § 20 (Fig. 20, Pl. IV.). It is 56 feet long, or 1.4 times the height. The anchorage-rods are 28 feet long over all. The upper chord of the 328-foot span is curved, the depth in the middle being 16 feet, centre to centre; so that the bridge looks like a suspension-bridge, which was specified by the government. The middle span is 64 feet long, and its end-posts are connected with the cantilevers by adjustable bars, which shall be without strains under any load, and only would come into action if a violent wind were acting from below. Expansion can take place at the ends of the whole bridge. The small middle span was the outcome of many considerations. The bridge is too short and the movable load too small to show the economic features of the cantilever type. It was also intended to use the bottom-chords as stringers carrying wooden floor-beams;

and it was desirable to utilize the bottom-chords, which did not admit of being reduced in section, as the chords of horizontal bowstrings acting against wind-pressure.

The structure was carefully calculated to stand wind-pressure of 41 pounds per square foot of surface, supposed to strike the floor at the most dangerous angle from underneath. Round and closed forms have been used where possible, so as to reduce both wind-pressure and weight.

The actual weights agreed with the estimated weights within one half of one per cent, and the cantilevers are one third heavier per lineal foot at the towers than at their ends. Thereby the maximum moment arising from the permanent load of a cantilever is reduced by five per cent.

The bridge when fully loaded will be strained to five tons per square inch, of which just about one half is due to the movable load. The maximum deflection, which was determined graphically in combination with a tabular mode of calculation, was found to be 3.7 inches. All members were considered—anchor-bolts, tower-posts, diagonals, posts, and chords. Commencing at one anchorage, the horizontal and the vertical displacements of each point were determined.

The deflection of 3.7 inches, being at the rate of $\frac{1}{1061}$ part of the span, is not a good feature of the system, but it would have been less if the middle span had been 160 feet instead of 64.

The weight of the superstructure, with exclusion of the piers above the floor, is only just equal to what it would have been if an ordinary Whipple truss of 328 feet, with two small side spans, could have been built at that locality.

In reality its weight would have been greater had not extreme care been used regardless of price towards reduction of the weight of iron. For this purpose twelve $\frac{3}{4}$-inch steel ropes were used to make the chords for horizontal

wind-bracing, and the posts, the suspenders, and the top-chords of the middle span were welded tubes.

The cantilevers, moreover, required costly anchorage-masonry, which, contrary to the information furnished for the design, was found indispensable on both sides of the river.

Adding the iron towers, 61 feet high, of which 41 feet must be considered to belong to the superstructure, the weight of the metal of this bridge, in accordance with theory, is certainly greater than would have been needed for a Whipple bridge.

Lately Mr. C. C. Schneider has had the opportunity to design a continuous truss-bridge, with independent middle span, across the Niagara River. The two side spans, being too short to balance the live loads on the middle span, are provided with vertical anchorage. In similar instances in Europe, boxes with stones were used as counter-weights of ordinary continuous girders with short side spans.

A cantilever-bridge, such as defined by us, the new Niagara bridge is not.

But owing to the great height of the floor of this bridge above the water, the system * chosen by Mr. Schneider, which admitted of erection of the main span without false-works, was the proper plan to be adopted at that locality, and probably this system there was also more economical than would have been an arch-bridge.

However, we should have suggested a longer independent span; for instance, one equal to three eighths or one half of the length of the main span.

The greatest work in bridge-engineering, considering its size and its unprecedented difficulties as regards erection, when finished will be the Forth bridge, with its two double-track cantilever-spans, each of 1700 feet.

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* See *Railroad Gazette*, March 1884.

Originally a suspension-bridge was intended for the locality. It was already given under contract when the Tay bridge fell down, whereupon the project was reconsidered, was found defective, and was abandoned.

After some time the project was resumed, and in *Engineering* the design for a cantilever-bridge was published, with illustrations, by Mr. Benjamin Baker, the author of a book on "Long-Span Bridges," where continuous girders with "varying economic depths" were recommended.

In September 1882 the author of that book read a paper before the British Association on the final "design unanimously agreed upon by all as the one to be recommended to the directors for adoption" (see *Engineering*, September 1 and 8, 1882).

"The size of a bridge is very commonly the popular standard by which the eminence of its engineer is measured."

"As the largest railway-bridge in this country [England], the Britannia bridge, has a span of 465 feet, and the Forth bridge a span of 1700 feet, the ratio there is 1 to 3.65. Hence, to enable any one to appreciate the size of the Forth bridge, we have merely to suggest the following simple rule-of-three sum: As a Grenadier Guardsman is to a new-born infant, so is the Forth bridge to the largest railway-bridge yet built in this country. Bridges a few feet larger in span than the Britannia have been built elsewhere, but they are baby-bridges after all."

After these introductory words of the eminent creator of the design of the largest railway-bridge, we shall now describe its principal features, reference being had to the diagrams on Plate VI.

The two 1700-foot spans will have a common economic anchorage, 270 feet long, on an island, and the weight of each cantilever there forms the counter-weight for the other.

There are fixed horizontal back-struts below, and diagonal stays to bring home additional strain in case one cantilever-arm and adjacent middle span were filled with movable load.

The other anchorages (see Pl. VII.) of the 1700-foot spans are made in a manner which should not be imitated.

Wide piers with horizontal back-struts, horizontal top-chords, and with diagonals are formed.

These piers are 155 feet long and 350 feet high. Then outside spans of 675 feet length are added, of which one is entirely upon dry land, and the other stretches over ground which, to judge from the diagrams published, must be dry, or very nearly dry, at low-water mark.

The anchorage-spans are much too long. The minimum strain-length for the height of 350 feet is obtained for a length of 304 feet, whilst the anchorage-length of the Forth bridge is to be 675 feet plus 155 feet, causing an excess of 64 per cent over the minimum strain-length of anchorage with inclined back-struts.

The cantilevers will be of varying depths.

The designer, on the strength of rough calculations leading to absurd results (see his "Long-Span Bridges"), believes that variation of depth is an essential element of economy of a continuous girder of varying depth, and therefore has made the end-depth very small and the depth at the pier very great, so as to secure the greatest possible field for variations.

The top-chords of the cantilevers will be in straight lines, and, according to the plan unanimously agreed upon, they would be 6 feet closer together at their ends than at the towers. They will consist of open truncated pyramids, latticed in four surfaces, and with the basis (10-foot sides) at the towers.

The bottom-chords and compressional diagonals are to be round tubes.*

Also, the bottom-chords are to be of variable diameters, curved, and lying in inclined surfaces.

Accordingly, the webs of the cantilevers would lie in warped surfaces.

As regards the towers, the first design exhibited six pairs of slanting end- or tower-posts about 360 feet long, and projecting about a hundred feet. At each of the outside anchorages two of these posts were to meet in a point. The design unanimously adopted shows the towers each occupying 155 feet of the length of the bridge, such as mentioned above.

But whereas in the first design these posts and those on the island were about 200 feet farther apart at their tops, now they are to be 10 feet farther apart below.

In the first design all other cantilever-posts were vertical in elevation, with diagonals of single intersection; now the posts are to be inclined and double intersection is introduced.

In the first design the panels were equally long; in the last design the cantilevers are arranged with panels of varying lengths, with a maximum stretch of about 150 feet.

In the first design the middle spans were 500 feet long; in the adopted design they are reduced to 350 feet, and—probably for the sake of symmetry—the anchorages of the new design are made longer still than those of the first design.

At first the middle spans were arranged with vertical posts; now triangular web-bracing is introduced.

Great stress is laid on the "resolute" step of adopting a

* The difficulty of making joints with these round tubes seems to have been too great, for we learn that the diagonal tubes are now to be square with rounded corners. It must be a difficult piece of work to build them to correct dimensions.

greatest depth of one half of the length of a cantilever-arm. But this principle goes beyond the practical limit of requirement; it is not applied to the middle span, where it would be much more nearly in place, and it is ill applied to cantilevers. For it is known that, on the contrary, cantilevers have the advantage of requiring comparatively smaller depths than ordinary truss-bridges because their moments, on the average, are smaller.

At the towers the bottom-chords are spread to a basis of 132 feet. At the top two corresponding tower-posts are to be 33 feet apart.

If the large spans of the Forth bridge had been divided according to the requirements of greatest economy, the *uniform* width of the bridge could have been reduced to 84 feet without causing on the average greater chord-strains from wind than now will be obtained with a maximum width of 132 feet.

Had the spans been divided as suggested in § 23, the top-chords would have been *horizontal*, and the greatest depth would have been 100 *feet less*. These horizontal top-chords, with trusses equally far apart and in perpendicular position, would have offered facilities during erection. The top-chords, which in the design adopted are so close together as to be useless as chords for a horizontal top-bracing, would have assisted in neutralizing the wind-pressures, and the bracing between parallel chords 84 feet apart would have been both stiff and economical.

In consideration of the use of steel (see Appendix No. 1),*

* From the success of steel for rails, boilers, and ships, it must not be concluded at once that this material must also be superior to *good* iron for large riveted skeleton-bridges. The superiority of steel rails lies in the hardness of the material and in its having no welds. In open bridge-work members with reliable longitudinal fibres are needed, and the lateral welds of these fibres are of inferior importance. In boiler and ship work thin sheets are riveted together

and the huge dimensions of a structure weighing 13½ tons per lineal foot at the towers, and considering the length of panels, the fantastical irregularity of form, the mixture of hollow round and of open square sections, variable in diameters, engineers who have practically learned how to design working details of bridge-work will doubt very much whether reliable riveted connections can be executed, more especially for the tensile joints.

They will not be convinced if they are told that "such difficulties vanish, as usual, upon being grappled with." The novelty of warped web-surfaces, for instance, was not of this kind, for it is already abandoned.

That this not very mature design, besides presenting no features of practical eminence or any show of previous experience or analytical training of an expert, must necessarily lead to a very heavy and to a very expensive, and yet by no means comparatively strong, structure, results from its uneconomical proportions.

Notwithstanding the economy offered by the central anchorage, the two complete 1700-foot spans are estimated to require 42,000 tons of steel.

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with the utmost diffusion of strains. But in modern bridge-work material lines, and not material surfaces, are formed. Not material lines are riveted together, but at material points—namely, the joints—enormous forces are concentrated which must be transferred and must be spread again over the area of the next member or members. And when enormous and untried spans are reached the question arises whether the practicable limit of riveted connections has not been reached too, and especially the influence of secondary strains has to be encountered, which strains are thrown on comparatively small rivets at considerable distances from the gravity-lines of the members, and are greatest at the joints. Before entering into the experiment of building riveted steel cantilevers, some 1500 feet long, experiments on complicated riveted connections of nearly corresponding size should have been made both for compression and for tension. Experiments like those which the designer of the Forth bridge makes and usually quotes, "of which there were not a few"—for instance, the experiment with a 4-inch ordinary stove-pipe 2 feet long—are not sufficiently scientific.

When the gigantic undertaking has been successfully finished there will be great honor and merit due to the contractors; for their task, great under any circumstances, has been made as difficult as possible.

§ 25. The Stiffening Girders of Arch- and Suspension-Bridges.

The stiffening girders of arch- and suspension-bridges will here be treated in such a manner that one set of simple formulæ will answer for both.

In particular it is not intended to treat the arch as an unhinged curved beam. An arch with three hinges is already sufficiently complicated as to form. Economy results from the proper proportions of depth to span, from the depth of the stiffening trusses, from the reduced number of panels, and from the design in detail. Fixing the points of reversion of moments by hinges adds to simplicity and to certainty of calculation of the strains,* and thereby already admits of higher specific strains, and this means economy.

Consider a flexible arch or a catenary which is separated from its stiffening girders in such a manner that the vertical connecting members (posts or suspenders) are hinged at their ends. Under these circumstances the stiffening girders have the office of distributing a part of the load P over the arch or over the catenary according to a certain law, due to the form of the latter. The curves of the arches or of the catenaries at each panel-point change the direction of their tangents. The horizontal reaction Q (only vertical forces—loads—being supposed as exterior forces) is a constant value throughout the length of the arch.

* We learn that of the new arch-bridge at Coblenz, some diagonals sheared off the rivets by which they were attached to the chords of the stiffened ribs.

If the curve of the arch or of the catenary is represented by the equation $y = F(x)$, the product

$$Q \cdot \frac{dy_1}{dx_1}$$

will represent the shearing force for the point with the co-ordinates $x_1 y_1$. The expression

$$Q \cdot \frac{d^2y_1}{dx_1^2}$$

will be the change of the shearing force at that point. These values would be exactly correct if there existed an infinite number of panels.

If the number of panels is considered as limited, differences must be substituted for differentials.

The change of the shearing force at the panel-point equals the force π_{x_1}, which at the point $x_1 y_1$ is caused by the strain in the suspender (or the post in case of an arch) at the same point.

The form of the catenary is optional, provided the stiffening girder and the suspenders are proportioned accordingly.

It may, for instance, seem rational to make π_x a constant value, and the conditions for so doing are:

1. The stiffening girder must be so strongly designed that the deflections and consequently the deformations of the flexible arch or of the catenary are so small that they can be entirely neglected.

2. The value $\frac{\Delta^2 y}{\Delta x^2}$ must be a constant quantity for equally long distances or panels Δx.

It is known that the parabola is the curve which fulfils the second condition, and its equation (see Fig. 22, Plate IV.) will be

$$x^2 = \frac{a^2}{h}(y - s).$$

We suppose that the girder CD originally does not touch the bearing-plates C and D. If the movable load arrives at C, the girder CD deflects to a small extent and finds its bearing. At the other end the girder will remain free.

The parabolic catenary acts as a machine by which a part of P is equally distributed over all suspenders. We have now

$$m.d.P = [1 + 2 + 3 + \ldots + (2n - 1)].\pi.d;$$

$$\pi = P.\frac{m}{n(2n - 1)}; \qquad C = P.\frac{x}{a}.$$

The qualities of design (Fig. 22) of a suspension-bridge stiffened by an unhinged girder will now be more closely examined.

The anchorage is already treated, and the catenary has the same strain-length as the bow of the parabolic bowstring-girder.

The average length of the suspender is $\left(\frac{h}{3} + s\right)$.

If p_1 is the permanent load less cables, and if p_0 is the movable load per lineal foot, the strain-length of all suspenders will be

$$\frac{2a}{3}(P_1 + P_0)(h + 3s).$$

The average maximum moment will be found to be $0.132p_0.a^2$, and the maximum moment of all is $0.181p_0a^2$.

The average maximum shearing force is $0.2425p_0.a$.

And since the same forces will cause diagonals of the same size to be placed in the other direction if the load comes from the other end of the bridge, either the diagonals must be built compressive as well as tensile or else counter-

diagonals will be necessary. In the latter instance the posts will have the average pressure of $0.33 p_0 . a$.

The maximum moments and shearing forces are shown by Fig. 23, Plate IV.

Adopting a stiffening girder of the depth h_1, vertical posts and tensile diagonals in panels of the length $\frac{a}{n}$, we obtain the strain-lengths as follows:

For the chords, $\quad 0.528 p_0 . \frac{a^2}{h_1};$

For the posts, $\quad (2n + 1) . 0.33 p_0 . a h_1;$

For the diagonals, $\quad 2n . 0.485 . p_0 a . \frac{h_1^2 + d^2}{h_1}.$

Hence for the web and chords,

$$Sl = p_0 \frac{a^2}{h_1} \left(0.528 + \frac{0.97}{n}\right) + p_0 a h_1 (0.33 + 1.628 n).$$

For $n = 10$, $h_1 = 0.194 . a$ will give the minimum $6.44 p_0 a^2$.

The girders of suspension-bridges must have both chords arranged for tension and for compression on account of their being also the chords for horizontal wind-trusses. A suspended girder without central hinge would also sustain flexures caused by changes of temperature, which in climates where these changes are great would require the adoption of more flexible low and consequently more expensive stiffening girders, unless by some special arrangement, such as equalizing levers or self-adjusting continuous suspenders passing over pulleys, the tensions of the suspenders were equalized.

§ 26. Stiffening Girder Hinged in the Centre.

The difficulty of additional strains arising from changing temperatures can be better avoided by the introduction of the central hinge. We owe this improvement to the engineer Koepke of Hanover (1860). It was, though not formally expressed, already implied in a report of a commission of Prussian engineers on Roebling's suspension-bridge over the Niagara River.

We use the denotations of Fig. 22. In E the half-girders are hinged so as to be free from moments of flexure. Two reacting forces, C and D, will now appear, and if the load P lies between C and E, the reaction D will be negative or it will act downwards. Again, if P acts between E and D, the reaction C will be negative. It is again supposed that there exists some little play at C and D which is neglected in calculation.

The girders are supposed to be absolutely stiff, and the catenary to be a parabola. The equations

$$C + 2a.\pi - D = P,$$

$$D.a = \pi\frac{a^2}{2} \qquad \text{and} \qquad C.a + \pi\frac{a^2}{2} = P.x$$

give

$$\pi = P.\frac{b}{a^2}; \quad C = P.\frac{3x - a}{2a}; \quad D = P.\frac{a - x}{2a} = P.\frac{b}{2a}. \qquad (41)$$

If the load P were replaced by $p_0.dx$, and if a uniform load $b.p_0$ were supposed to advance from C towards E covering the length b, there would be obtained

$$\pi = p_0.\frac{b^2}{2a^2}; \quad C = p_0.b.\frac{4a - 3b}{4a}; \quad D = p_0.\frac{b^2}{4a}. \qquad (42)$$

It is seen that the force π in the suspenders is now double of what it was for the unhinged beam, from which it is concluded that the flexures of the half-girders will be smaller.

For $z < b$ the moment is

$$M_z = C \, . \, z - (p_0 - \pi)\frac{z^2}{2},$$

which is a maximum for $z = \dfrac{C}{p_0 - \pi} = \dfrac{ab}{2} \, . \, \dfrac{4a - 3b}{2a^2 - b^2}$, and the maximum M_z is

$$= p_0 \, . \, \frac{b^2}{16} \, . \, \frac{(4a - 3b)^2}{2a^2 - b^2}.$$

For $b = 0.79a$ or $z = 0.467a$ the greatest moment of all is found $= p_0 \, . \, \dfrac{a^2}{13.23}$.

For a fixed z the moment becomes a maximum if $b = \dfrac{2a^2}{3a - z}$ and the moment is $\dfrac{pz}{2}\left(\dfrac{b}{2} - z\right)$.

$z =$	$\frac{a}{8}$	$\frac{a}{4}$	$\frac{3a}{8}$	$\frac{a}{2}$	$\frac{5a}{8}$	$\frac{3a}{4}$	$\frac{7a}{8}$	a	(42*a*)
$\frac{b}{a} =$	0.7	0.727	0.762	0.8	0.842	0.888	0.94	1	
$M = p_0a^2 \times$	0.036	0.060	0.073	0.075	0.067	0.052	0.028	0	

The average moment is $\dfrac{p_0a^2}{20.8}$.

If the load p_0b advances beyond E, or if b becomes greater than a, there will appear in the other half-span moments absolutely as great as those of the preceding table, but *negative*.

Each piece will be strained to a maximum tension, and also

to a maximum pressure of the same absolute value. The chord-joints therefore must be built very carefully. It will be advisable to use long panels so that enough section will be given which is to provide against crippling. Thus the material otherwise lost by rivet-holes and that portion of material which has to be added on account of reversion of strains (see § 3) is utilized.

For the shearing forces of the girders there is found, (*a*) for the loaded part $z < b$,

$$C - (p_0 - \pi) z = p_0 \left[1 - b\left(\frac{3}{4a} - \frac{z}{2a^2}\right) - z \right].$$

The length z being considered a constant, the shearing force becomes a maximum if $b = \dfrac{2a^2}{3a - 2z}$, and the maximum is $p_0 \left[\dfrac{a^2}{3a - 2z} - z \right]$.

$$\left.\begin{array}{llccccc} \text{For} & z = & 0 & \frac{a}{8} & \frac{a}{4} & \frac{3a}{8} & \frac{a}{2} \\ & \frac{b}{a} = & \frac{2}{3} & \frac{8}{11} & \frac{4}{5} & \frac{8}{9} & 1 \\ \text{Shearing force} = p_0 \,.\, a \times & & \frac{1}{3} & 0.24 & 0.15 & 0.07 & 0 \end{array}\right\} \quad (43)$$

And if the load advances beyond E, the same but ***negative*** forces are produced.

(*b*) Further, if a load is on one side of the bridge, there are already ***negative*** shearing forces on the other side, namely,

$$D - \pi \,.\, z' = p_0 \,.\, \frac{b^2}{4a}\left(1 - \frac{2z'}{a}\right),$$

where z' is the distance from the bearing D.

These strains are maxima for $b = a$, and we obtain for

$$\left.\begin{array}{lccccccccc} Z' = & 0 & \frac{a}{8} & \frac{a}{4} & \frac{3a}{8} & \frac{a}{2} & \frac{5a}{8} & \frac{3a}{4} & \frac{7a}{8} & a \\ \text{Shearing force} = -p_0 \,.\, a \times & 0.25 & 0.19 & 0.125 & 0.063 & 0 & 0.063 & 0.125 & 0.19 & 0.25 \end{array}\right\} \quad (44)$$

(*c*) If this calculation is continued for $z' > a$, but z either equal to $(2a - b)$ or smaller than $(2a - b)$, commencing again at C, after introduction of $z'' = 2a - z'$, we have

$$\text{Shearing force} = C - p_0 \,.\, b + \pi \,.\, z'',$$

and we obtain the minimum for $z'' = b$.

$$\left.\begin{array}{lccccccccc} \text{Minimum} = & \multicolumn{9}{l}{-\frac{3p_0 \,.\, a}{4}\left(\frac{b}{a}\right)^2 \,.\, \left(1 - \frac{2}{3} \,.\, \frac{b}{a}\right)} \\ \text{For} \quad b = & 0 & \frac{a}{8} & \frac{a}{4} & \frac{3a}{8} & \frac{a}{2} & \frac{5a}{8} & \frac{3a}{4} & \frac{7a}{8} & a \\ \text{Shearing force} = -p_0\, a \times & 0.01 & 0.039 & 0.08 & 0.125 & 0.175 & 0.210 & 0.240 & 0.25 & \end{array}\right\} \quad (45)$$

And the bridge having been filled and the train leaving it, the same but positive forces will be obtained.

The average of all shearing forces is $\pm\, 0.2p_0 \,.\, a$. (See Fig. 24, Pl. IV.)

Suppose now the girders to be designed with all diagonals at 60 degrees, and suppose that on account of reversion of strains 50 per cent additional area is given, we obtain

For the chords, $0.332p_0 \,.\, \frac{a^2}{d}$;

For the web, $1.386p_0 \,.\, a^2$.

$$Sl = 2p_0 \,.\, a^2\left(0.166\,\frac{a}{d} + 0.693\right). \quad . \quad . \quad . \quad . \quad (46)$$

For $d = \frac{a}{10}$, Sl becomes $4.7p_0 \,.\, a^2$ instead of $6.44p_0 a^2$ of the unhinged girder.

We find for the strain-length of a whole suspension-bridge with hinged girders as follows:

For the catenary, $(p_0 + p_1)\dfrac{a^2}{h}\left(1 + \dfrac{4h^2}{3a^2}\right)$;

For the suspenders, say $(p_0 + p_1)\dfrac{3ah}{4}$;

For the mainposts, $(p_0 + p_1)2a(h + h_1)$.

The moment which controls the anchorage is $(p_0 + p_1)\dfrac{a^2}{2}$.

If h_1 is assumed to be $\dfrac{h}{3}$, supposing also natural anchorage and anchor-bolts $\dfrac{h}{3}$ long, the minimum strain-length for two anchorages will be exactly $4(p_0 + p_1)a^2$.

The first three expressions

$$(p_0 + p_1)\,a^2\left(\frac{a}{h} + \frac{19}{4}\frac{h}{a}\right)$$

will be a minimum for $h = \dfrac{a}{2.3}$, namely, $4.6\,(p_0 + p_1)\,a^2$.

For the whole suspension-bridge there is

$$Sl = a^2\,(8.6\,.\,p_1 + 13.3\,.\,p_0).$$

This suspension-bridge is too deep. If $h = \dfrac{a}{4}$, $h_1 = \dfrac{a}{7}$, the minimum strain-length of anchorages will be $4.84(p_0 + p_1)\,a^2$ for $z = 0.605\,a$, and the catenary suspenders and mainposts become $5.2a^2(p_0 + p_1)$; and finally,

$$Sl = a^2(10.04p_1 + 14.74p_0),$$

which is less favorable than the strain-length of a cantilever-bridge.

But the advantage of a suspension-bridge consists in these points: Its cables, suspenders, and anchorage-cables can be made of steel wire, which admits of tensions at least double as great as those of a cantilever-bridge chord. It can be more easily erected than the latter, and by the distortion of the catenary under non-uniform loads the moments and shearing forces are less than calculated under the supposition of absolutely rigid stiffening girders. If this reduction amounts to 20 per cent of the strain-length of the stiffening girders, the strain-length of the parts of a suspension-bridge $\left(\text{with } h = \frac{a}{4}\right)$ between the piers may be considered as equivalent to

$$a^2(2.6p_1 + 6.36p_0),$$

which is a very small value and shows that p_1 must be comparatively very small.

In the sequel we shall enter more deeply into the theory of stiffened suspension-bridges.

§ 27. Application of the Central Hinge to Arches.

The formulæ of the last section are applicable not only to the stiffening construction of separate girders of suspension-bridges, but also to the spandrel-braced arch and suspension-bridge, and they can also be used if the arch or the catenary is split in two parts between which suitable bracing is arranged.

The strains in these instances result from the superposition of the strains of the girder and the strains arising from the horizontal reaction Q.

With arch-bridges at least one chord of the stiffening girder will be the arch itself. If the floor lie above the arch, posts will be needed to carry the floor. One set of chords being already given, the other chords are placed in the surface of the floor, and these constitute also the chords for the horizontal wind-bracing of the floor.

Even if the floor of an unusually large arch-bridge has to be suspended, spandrel bearing with top-chords above may be desirable, because of the facility of erection by anchorage.

If arches of the stiffened-rib type be chosen, there will be needed two sets of posts, one between the ribs, the other to carry the floor. The depth of the arch must be made sufficiently small, the distance between the ribs and the width of the bridge must be made sufficiently great, so that there may always remain some pressure in one rib, and that there may not be too much pressure in the rib diagonally opposite for any combination of loads and horizontal wind-pressure.

For the spandrel-braced arch no such conditions are imposed upon its depth, and this is one of the reasons why, in combination with facility of erection and greater stiffness under wind, the spandrel-braced arch may be preferred.

In order to illustrate the economic qualities of the arch we suppose spandrel bracing in 16 panels, with depth of arch equal to one seventh of the span $2a$. The depth at the crown is assumed to be $\frac{a}{32}$.

For the arch itself the strain-length is found

$$3.881a^2(p_0 + p_1).$$

Next we form the maximum moments and divide each such moment by its corresponding depth of truss.

Panel-points.. ..	0	1	2	3	4	5	6	7	8
Moments = $pa^2 \times$	0	0.036	0 060	0.073	0.076	0.067	0.053	0.030	0
Heights = a $\times$	0.32	0.25	0.19	0.14	0.10	0.07	0.05	0.04	0.03
$\dfrac{M}{p_0 \,.\, h \,.\, a}$	0	0.144	0 32	0.52	0.76	0.96	1.06	0.75	0
Differences.......		0.144	0.176	0.20	0.24	0.20	0.10	− 0.31	− 0.75
Secants for posts..	2	1.53	1.15	0.83	0.56	0.4	0.3	0.2[illegible]	
Post-strains = $pa \times$	0.288	0.28	0.23	0.20	0.112	0.040	0.093	0.187	
Post-lengths = $a \times$	0.32	0.25	0.19	0.14	0.10	0.07	0.05	0.04	0.03
Strain-lengths of posts = $p_0 a^2 \times$	0.092	0.070	0.044	0.028	0.012	0.028	0.047	0.075	0.056

For all posts together, $0.85 p_0 \,.\, a^2$.

Again:

Differences......	0.144	0.176	0.20	0.24	0.20	0.10	0.31	0.75
Squares of secants of diaognals...............	7.56 5.06	5 06 3.31	3.31 3.06	3.06 1.61	1.61 1.28	1.28 1.21	1.21 1.10	1.1 1.1
Strain-lengths = $p_0 ad \times$	1.82	1.47	1.27	1.12	0.58	0.25	0.72	1.65

Strain-length from all diagonals, $2.22 p_0 \,.\, a^2$.

For the top-chords (specifying as before the admissible maximum strain as only two thirds of the usual maximum) the strain-length $1.58 p_0 \,.\, a^2$ will be found.

From permanent load there is still the strain-length $0.13 p_1 a^2$ for the posts, half of p_1 being supposed to lie in the arch.

Hence total strain-length of arch and girders:

$$Sl = \Big[(3.88 + 0.13)p_1 + (3.88 + 0.85 + 1.58 + 2.22)p_0\Big] \,.\, a^2$$

$$= (4.01 p_1 + 8.53 p_0)\, a^2.$$

For different proportions of p_0 to p_1 different values are obtained.

$$\left.\begin{array}{llllllllll}
\dfrac{p_1}{p_0 + p_1} = \dfrac{p_1}{p} = & \dfrac{1}{6} & \dfrac{1}{5} & \dfrac{1}{4} & \dfrac{1}{3} & \dfrac{1}{2} & \dfrac{2}{3} & \dfrac{3}{4} & \dfrac{4}{5} \\[2ex]
\dfrac{p_0}{p_0 + p_1} = \dfrac{p_0}{p} = & \dfrac{5}{6} & \dfrac{4}{5} & \dfrac{3}{4} & \dfrac{2}{3} & \dfrac{1}{2} & \dfrac{1}{3} & \dfrac{1}{4} & \dfrac{1}{5} \\[2ex]
Sl = pa^2 \times & 7.78 & 7.62 & 7.4 & 7.03 & 6.27 & 5.51 & 5.14 & 4.91
\end{array}\right\} \quad (47)$$

Similar tables can be formed for any other depth of arch or any other number of panels.

A Whipple truss, depth of span equal to one seventh, with 14 panels, furnishes $7.32pa^2$, and for 10 panels only $7.02pa^2$. Table 47 therefore proves that if the permanent load is at least one third of the movable load, the arch is as good as the best Whipple truss. The greater the permanent load in proportion to the movable load the better the arch will be, provided there is sufficient head-room to make it deep.

What is especially valuable in the arch is the comparatively great number of panels; this quality in connection with the very low strain-lengths makes it an excellent form of structure for spans of extraordinary length.

Arches require also stone abutments. But if these can be established on dry land, or if the arch is thrown across a ravine with steep slopes, the material for abutments may cost less than the piers of a girder-bridge.

The arch is in its proper place if a deck-bridge is admissible. If the span is very great and the floor is to be suspended, another form will be more appropriate, which will be treated in the next section.

§ 28. Cantilever Arch-Bridges with Five Hinges.

It is possible to reduce considerably the strain-length of the stiffening construction of a large spandrel-braced arch by combining with it the good features of the cantilever plan.

The arch (Fig. 25, Pl. IV.) consists of a central hinged arch of the length $2b$, and of two cantilevers, c. They will be anchored horizontally if the locality will permit it, and the material of the anchors, d, will make part of the chords for the small side-spans. Otherwise the anchorage will be executed in one of the best manners previously analyzed.

If the whole arch were uniformly loaded, only the parabolic arch itself would receive strains, while the web-diagonals and the top-chords, as well as the cantilevers, must be free of strains.

If the central arch $2b$ were filled with movable load (or if it were loaded by $z \, . \, p_0$—see figure), both anchorages (or one anchorage) would have to sustain thrust in the upper chords, and the cantilever-braces would receive strains corresponding with this thrust. But if a cantilever alone were filled with movable load, this cantilever (or also, if the central span were loaded by $z \, . \, p_0$, the contiguous cantilever) and anchorage would be exerted as in case of ordinary cantilevers.

The five hinges A, B, C, B, A will provide for expansion. The moments of flexure as caused by wind are small, because the horizontal girders of which the arches form the chords enjoy the property of continuity.

There are two methods of avoiding compression in the back-stays of the anchorage:

a. By loading purposely and permanently the cantilevers so as to overcome the greatest possible compression arising from the movable load.

b. By placing the points B so deep that the direction of the last arch-piece of the middle arch, BCB, does *not pass above* the points A.

This may be done either by making AB a straight line coinciding with the tangent to the middle span in B, or by using line m or line n of Fig. 26, Pl. IV. Line AmB will be chosen if a steep tangential angle at A is required, and AnB will be preferred if suitable natural abutment exists in the line nA.

The tangent Bf of the arch at B passing *below* A, a moment of downward flexure of the cantilever is produced which will neutralize the negative moment caused by the movable load on the middle arch.

The half-spans are to be divided in such points *D* that the material required for the cantilevers is just sufficient to admit of erection without false-works. The erection being finished, the top-chords are separated at the chosen points *D*. The ingenious device of sand-boxes of the French engineer Baudemoulin (1849) may be used for this purpose.

It is obvious that the moments of flexure of the middle arch will be considerably less than they would have been without the cantilevers being separated. The top-chords, posts, and diagonals will be lighter. Thus the central arch will be very light, especially if by the arrangements shown by Fig. 26 the middle arch becomes much deeper than otherwise it would have been.

The arch with five hinges also enjoys the advantage of an increased number of panels in combination with diminished strain-lengths. The depths of the cantilevers, *ABDD* (Fig. 26), are very considerable and vary comparatively very little.

The whole construction also enjoys the advantage of concentration of masses near the supports; and since the material of such a bridge is smaller between the mainposts than that of any design previously known, it will be excellently adapted for very great spans.*

The cantilevers of very great spans would be arranged with manifold systems of diagonals.

It is not necessary that the last diagonals should all join in the points *D*. They may receive those directions which produce a minimum of strain-length, and separate anchorages may be arranged if found convenient. Those diagonals should be connected with the towers *AD* by means of standing or hanging pendulums, and the vertical forces should be uniformly distributed over the several ribs forming the towers or posts *AD*.

* Cantilever-arches were invented and patented by the author in the year 1873.

By separate anchorages, though they may be found somewhat more costly than one single anchorage further behind, erection may be facilitated. Indeed in case of very great spans decentralization may become very desirable.

§ 29. On the Arrangement of the Web-Systems of Stiffening Girders of Arch- and Suspension-Bridges.

If a design is of such a nature that the maximum strains of the chords happen for another position of the movable load than that for which the web-strains would become maximum, other rules for the best position of the web-members must be used than those resulting from §§ 18 and 19.

In such instances web-members must be treated separately, no matter what the strains of the chords may be.

The maximum differences of the values $\frac{M}{h}$ in a panel of the parabolic arch with three hinges are, one positive and the other negative, and their absolute values are equal for each panel. These differences vary in different panels. If, therefore, triangular web-bracing is adopted, each diagonal will sustain strains varying between equally great positive and negative maxima.

We shall now examine different modifications of arrangements of web-members.

We assume non-parallel chords, the panel-heights to be h_1 and h_2, the moments to be M_1 and M_2, the shearing forces S_1 and S_2. That part of S_1 which is not absorbed by the inclined chords is denoted by P. It is equal to $\frac{M_2 h_1 - M_1 . h_2}{h_1 l}$. (See formula 25.)

In a similar manner $p_2 = \frac{p_1 . h_1}{h_2}$. (See formula 27.)

The horizontal component of the diagonal force is

$$P_h = P_1 \cdot \frac{d}{h_2} = P_2 \cdot \frac{d}{h_1}$$

if, as shown by Figs. 27 and 28, Pl. IV., the line d is horizontal so that it becomes equal to l, whereupon again

$$P_h = \frac{M_2 \cdot h_1 - M_1 \cdot h_2}{h_1 \cdot h_2} = \frac{M_2}{h_2} - \frac{M_1}{h_1},$$

such as indeed must be the case.

(*a*) First we shall examine the bracing of the spandrel-braced deck-arch. The top-chord CD (Fig. 27) is horizontal.

(1*a*) Construction of posts h_1 and h_2 and tensile diagonals AC and BD. We find

$$Sl = \frac{P}{h}\left(2d + \frac{h_1^2 + h_2^2 + h_1 \cdot h_2}{d}\right). \quad . \quad . \quad . \quad (48)$$

(2*a*) We now examine under what condition pure triangular bracing will yield a minimum of strain-length.

We draw an imaginary perpendicular line EG, and denote by ε the distance of G from the centre F of line $CD = d$.

$$GE = \frac{h_1 + h_2}{2} - \frac{\varepsilon}{d} \cdot (h_2 - h_1)$$

and

$$Sl = \frac{\overline{CE}^2 + \overline{DE}^2}{d} \cdot P_h$$

$$= \frac{P_h}{d}\left[\left(\frac{d}{2} + \varepsilon\right)^2 + \left(\frac{d}{2} - \varepsilon\right)^2 + 2\left(\frac{h_1 + h_2}{2} - \frac{h_1 - h_2}{d} \cdot \varepsilon\right)^2\right].$$

This expression becomes a minimum for

$$\varepsilon = \frac{d}{2} \cdot \frac{h_2^2 - h_1^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The point E is so situated that angle CED is as nearly as possible a right angle.

If CED happens to be just 90 degrees, $CE^2 + DE^2$ will equal d^2, and there will be

$$Sl = P_h \, . \, d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

For $h_2 = h_1$, or for parallel chords, ε will be nothing and CE will equal DE.

If angle CED is greater than 90 degrees, Sl will be less than $P_h \, . \, d$; but it must be remembered that $P_h = P_1 \, . \, \frac{d}{h^2}$, or that if angle CED is great, P_h must also be great.

Fig. 27 shows that the strain which acts in the longer diagonal is greater than the strain in the shorter diagonal. This is not favorable because of the material to be added for stiffness under pressure.

(3*a*) If CE is made equal to DE, both diagonals are equally strained and of equal area, and there will be

$$Sl = \frac{P_h}{2}\left(d + \frac{(h_1 + h_2)^2}{d}\right). \quad . \quad . \quad . \quad . \quad . \quad (51)$$

If we add 50 per cent for the reversion of strains, and thereupon compare (51) with (48), we find a difference in favor of the arrangement of CE equal to DE, amounting to

$$\frac{P_h}{4}\left(5d + \frac{(h_2 - h_1)^2}{d}\right).$$

(*b*) It remains to treat the bracing of the spandrel-braced arch with suspended floor (Fig. 28).

(1*b*) is the same as (1*a*).

(2*b*) The strain-length of the diagonals *AE* and *BE* is to be made a minimum, or $\frac{x^2 + v \cdot y}{BF}$ is to be made a minimum.

The unknown length $\rho = OE$ being introduced, we find

$$x = \rho \cdot \frac{a}{a+e} = \rho \cdot \frac{h_2 - h_1}{h_2},$$

$$v = y \cdot \frac{h_1}{h_2},$$

$$Sl = P_h \cdot \frac{h_2 \cdot x^2 + h_1 \cdot y^2}{\rho(h_2 - h_1)};$$

and since $x^2 = e^2 + \rho^2 - 2\rho f$,

$$y^2 = (e+a)^2 + \rho^2 - 2\rho(f + d),$$

$$Sl = P_h \cdot \frac{(e^2 + \rho^2 - 2\rho f)h_2 + [(e+a)^2 + \rho^2 - 2\rho(f+d)]h_1}{\rho(h_2 - h_1)}.$$

Hence
$$\frac{h_2 \cdot e^2 + h_1(e+a)^2}{\rho} + (h_1 + h_2)\rho$$

must be made a minimum. Or, since $h_2 \cdot e = h_1(e+a)$, the expression $\frac{e^2 \cdot h_2}{\rho h_1} + \rho$ must be made a minimum; or

$$\rho \text{ must be } e\sqrt{\frac{h_2}{h_1}}.$$

The minimum itself will be

$$Sl \text{ min.} = 2P_h \cdot \frac{h_1 \cdot h_2}{(h_2 - h_1)^2}\left(a \cdot \frac{h_1 + h_2}{\sqrt{h_1 \cdot h_2}} - 2d\right). \qquad (52)$$

For $h_1 = h_2$ the chords will be parallel and Sl will assume the form $\frac{0}{0}$, but E will be in the centre of d.

Instead of ρ the value of line CE may be determined,

which is $$= \frac{h_1}{h_2 - h_1}\left(a\sqrt{\frac{h_2}{h_1}} - d\right).$$

For $h_1 = h_2$ this expression receives the true value $\frac{d}{2}$. (53

(3*b*) If E is taken in the centre of CD, we shall have

$$Sl = P_h \cdot \frac{d^2 + 4h_1 \cdot h_2}{2d}; \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and if also here the maximum admissible strain per square unit is lowered by one third of its value, the difference

$$P_h \cdot \left(\frac{5d}{4} + \frac{(h_2 - h_1)^2}{d}\right)$$

is found in favor of this arrangement over (1*b*) or (1*a*).

(4*b*) The two diagonals are made equally long:

CE will be found $= \frac{d}{2} + \frac{h_2^2 - h_1^2}{2d}$, or $\varepsilon = \frac{h_2^2 - h_1^2}{2d}$;

$$z = \frac{h_2 - h_1}{h_2}\left(f + \frac{d}{2} + \varepsilon\right) = \frac{a^2}{2h_2 \cdot d}(h_1 + h_2);$$

$$Sl = P_h \cdot \frac{x^2}{z}\left(1 + \frac{h_1}{h_2}\right) = P_h \cdot \frac{a^2 + 4h_1h_2}{2d}. \quad . \quad . \quad (55)$$

For $h_1 = h_2$ or $a = d$ there will be found

$$Sl = P_h \cdot \frac{d^2 + 4h^2}{2d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

(5*b*) If it were asked to make the strain-length of one diagonal equal to the strain-length of the other, the corresponding point, E, could be found by a geometrical construction, as follows:

Divide AB in a point, K, so that $AK : KB = \sqrt{h_2} : \sqrt{h_1}$; find the fourth harmonious point, g, for the points A, K, and B, and describe over Kg a circle of which Kg is the diameter. This circle will intersect the top-chord in the point E. The reason for this construction consists of the equation $x^2 = vy = y^2 . \frac{h_1}{h_2}$, giving $x : y = \sqrt{h_1} : \sqrt{h_2}$.

It will be noticed that Fig. 28 differs from Fig. 27 in this, that the longer diagonal carries a smaller strain than the shorter, whilst in Fig. 27 the longer diagonal bears also the greater strain.

In some panels of the stiffening girder of the arch the greatest chord-strains and the greatest force, P_h, result from the same position of movable load. In these panels the rules of § 19 remain in force.

But if the chord-sections are already fixed upon by the consideration of their strains during erection, the web-diagonals again will be independent of the chords. If a spandrel-braced arch with suspended floor were so arranged that also the top-chords are curved, the minimum strain-lengths of the braces can easily be found by drawing the lines $AD' = h_2'$ and $BC' = h_1'$ at right angles to the top-chord, whereupon the length of CE is again found with formula (2*b*) (52).

The value of P_h will be $P_1 . \frac{d}{h_2'}$.

The most favorable arrangement will be determined by trial, and it is possible to obtain a tolerably regular web-bracing of a spandrel-braced arch which comes very near to the minimum strain-length, and of which also the strength of long compression-members is duly considered.

It is thus seen that the design of spandrel-braced arches is connected with interesting geometrical problems.

Since such arch-bridges in the same frame combine the arch proper and the stiffening truss, great simplicity of system is desirable in order to reduce secondary or accidental strains.

Pin-jointed triangular bracing would be preferred on account of simplicity and economy. But the joints must be very carefully designed, with an abundance of bearing area for the pins, or (in case of fixed joints) for the bolts or rivets, which must fit accurately and must be very numerous.

§ 30. A Cantilever Arch-Bridge of 1700 feet Span.

Arches with five hinges exhibit all their good qualities if the roadway is above, and if the locality offers good natural abutments. But if the dimensions become very great, cantilever-arches may also be arranged with suspended floors. The arch with braced ribs and with suspended floor presents some considerable difficulty in finding room for the passage of trains through spaces left open between the unavoidable wind-braces. The cantilever-arch avoids these difficulties; it is also lighter between the abutments, and it can be erected without false-works in deep water. And if two lines of railroad are used, the suspended floors can be placed sufficiently far apart so that the floors and floor-girders furnish strong horizontal wind-trusses.

Plate VII. represents one of the 1700-foot spans of the Forth bridge treated as an arch with five hinges.

Two careful estimates were made, one independently of the other and with somewhat modified proportions. There was an interval of three months between these two estimates, the quantities of which agreed within one per cent.

The free span remaining is 1670 feet, and is divided into twenty equally long panels, twelve of which are occupied by the independent central arch. This arch of 1002 feet span is 144 feet deep. There are only two arch-frames, placed in vertical planes which are parallel and 83.5 feet apart. The top-chords at the crown are 35 feet above the hinge. For the sake of better appearance, a couple of light members bridge the gap above the hinge. These members are supposed to be free at one end.

The towers are 278 feet deep between the mathematical centre-points. Their real height above masonry would be 258 feet.

The top-chord line of the cantilevers and of the central arch is a parabola with a rise of 29 feet.

The cantilevers therefore are 278 feet deep at the abutments, and 168.5 feet deep at their ends. Their length is 349 feet.

The outside anchorage is supposed to consist of back-chains only, attached to rock if in existence, or else to masonry. The existence of rock is assumed, and the anchorage length is equal to $h\sqrt{2.50}$. (See § 20, *a.*)

Instead of this arrangement, with perhaps more economy on the whole, anchorage with inclined back-struts could have been used. (See § 20, *d.*)

The back-chains are shown to be supported by the columns of the trestle-bridge and by bowstring-trussing. The posts of the cantilevers are shown to be in perpendicular position. Also the end-posts of the 1002-foot arch are vertical ones, because this is the most economical arrangement at those places. The other web-members are inclined.

Those of the cantilevers are tensile diagonals, and are supposed to consist of pin-jointed links.

The rest of the web-members of the middle arch are tie-struts. They are not exactly in those positions which result

from application of formula (52), but are very nearly in those positions by which the sum of the strain-lengths of the two diagonals of a panel becomes a minimum.

The lengths CE (see § 28), taken from the upper points of the lines h', should be successively

	83	66.6	62.5	47.2	42.7	41.8 feet,
for which	$83\frac{1}{3}$	$69\frac{1}{2}$	58.3	49.9	44.3	41.5 feet

were adopted, so that the top-chord pieces gradually diminish in length towards the centre of the bridge.

Thus the strain-lengths of the diagonals of a panel are nearly equal (§ 28, 5*b*), and the areas of each two diagonals are nearly the same. The diagonals ending in the centre hinge are reinforced because of the wind-strains.

The floors of the arch are suspended, and they are 64 feet apart between their centres. The iron girders carrying the floor would be hinged in the centre of the arch, and would be continuous over six panels. They were calculated for a rolling load of 1.5 tons per foot of each track, and a maximum strain of 4.5 tons per square inch. Their permanent load is half a ton per foot of each track. The iron floors were assumed to consist of rolled beams carrying channels, placed lengthwise, spliced and properly connected by tapering bolts.

The floor-girders of the cantilevers are shown to be trussed, and to converge in suitable curves, so as to be joined into broad floors at the end-towers.

The pairs of continuous floor-girders of the arches would be connected by strong wind-bracing, consisting of lateral struts and tensile diagonals. The wind-bracing of the bottom-chords of the cantilevers would consist of diagonal tie-struts.

The web-members are supposed to be provided with strong well-fitting pins.

In order to draw a comparison between the design of the

Forth bridge as now under construction and the arch with five hinges, the same movable loads were supposed, namely, one ton for each track.

The wind-pressure assumed is one ton per foot of the bridge, while, calculating from data contained in the paper read in August, 1882, before the British Association, we found that only about three fourths of a ton was supposed by Mr. Baker.

In agreement with the specification for the Forth bridge, the maximum strains per square inch resulting from the loads and from wind are not exceeding 7.5 tons.

In order to make the frames sufficiently strong to bear the strains during erection, the top-chords of the cantilevers and the three contiguous chords of the arches are reinforced. These reinforcements are included in the estimate of weights.

The floors and their girders were first calculated; then the wind-bracing was estimated both for the arch and the cantilevers. For the 1002-foot arch the average weight of its lateral and transverse wind-braces was found to be 892 pounds per foot. For the whole bridge the weight was found to be $0.94l$. Thereupon the strains and weights of the other parts of the independent arch were ascertained. In determining these strains a combination of calculation and of graphic methods was used.

In a similar manner also the strains of the members of the cantilevers and of the tower-posts were ascertained.

The strains of the cantilevers were found as arising from the dead load of the central arch, the full movable load of the central arch, the permanent load of the cantilever, the movable load of the cantilever, the loads during erection (no floors attached), and the action of wind on the fully loaded bridge.

The maximum strains and corresponding net areas are shown on Plate VII.

The weight of the material of the central arch was found to be 2.05 tons per foot of each track.

The panel-loads of the structure itself of each cantilever were supposed to be 337, 282, 227, and 172 tons.

On the average, the cantilevers, exclusive of the towers, were found to weigh 3.72 tons per foot of each track. In these weights the rails and the planks upon which they are supposed to be placed are included, both amounting to 0.1 of a ton per foot of two tracks.

One complete 1700-foot span between the towers (without rails, etc.) contains metal	9,124 tons
One central anchorage and tower for two tracks.	688 "
One outside " " " "	1,196 "
One complete 1700-foot span..................	11,008 tons
For the viaduct 22 feet 6 inches wide in its floor, for its parapets, its trestle-piers 57 feet wide below, all entirely of puddled iron, calculated for a live load of 3 tons per foot, and wind 1000 pounds per foot of floor, the weight 1.2 tons per foot is found for two tracks, or for the viaduct 5330 less 3460 = 1870 feet..................................	2,244 tons
To which add another 1700-foot span	11,008 "
Or for 5330 feet of bridging..................	24,260 tons

For the same length of bridge the adopted plan of the Forth bridge requires 42,000 tons of steel.

Difference in favor of the cantilever-arch 17,740 tons, or 73 per cent of the weight required for the latter design.

But it must not be forgotten that this difference would be much reduced if the Forth bridge, instead of being treated as a continuous girder of varying depth, width, etc., were designated as a scientifically proportioned cantilever truss-bridge.

In our design, according to a rather rough estimate the eight abutments and the trestle-piers would require about 60,000 cubic yards of stone-work. If there were no natural anchorage, about 40,000 cubic yards of masonry would be needed for this purpose, the weight of which increased by mere ballast would give the necessary resistance during erection. In this case the metallic end-piers and end-anchorages would weigh about 10 per cent less.

The design now in process of execution is stated to require about 125,000 cubic yards of masonry for the whole bridge, 8084 feet long; and in the absence of special statements for the cubic contents of its twelve piers and of its counterweights at the ends of the 675-foot outside spans, it may be assumed that for the bridging of the great openings such as described about 110,000 cubic yards would be required.

At all events, there is no reason to doubt that also in this regard the plan adopted for the Forth bridge is not superior to the arch with five hinges. Besides, as already stated, anchorage with inclined back-struts could have been estimated on for the cantilever-arch, which would have saved some masonry without materially adding to the weight of iron or steel.

§ 31. Discussion of the Strain-Sheet and of the Weights of the 1700-foot Cantilever-Arch.

The diagram of strains of Plate VII. shows that under full load the horizontal thrust against each abutment is 4160 tons. The average total load between the piers, including train, rails, and road, is 3.735 tons per lineal foot of track.

Taking the mathematical height of the tower as lever, we find that the thrust of 4160 tons corresponds with the expression $0.43pa^2$. Hence by the concentration of weights

near the towers a reduction of 14 per cent of the maximum moment is secured.

But the strain-sheet also shows that the corresponding maximum horizontal pull at the top of the tower is not 4160 tons, such as would be the case if the structure were an ordinary cantilever-bridge. This tension is only 1681 tons, or 0.4 of the thrust below. Therefrom follows the valuable observation that the anchorage material and the anchorage masonry, if any were added at all, of the arch with five hinges is only *four tenths of that of an ordinary cantilever-bridge.*

The central arch has strain-lengths as follows:

For the top-chords (counted double, which is certainly a liberal allowance for its bearing tensions as well as pressures) .... $1.403 p_0 a^2$
For the web, under the same supposition .... $1.627 p_0 a^2$
For the arch itself .... $(p_0 + p_1) \cdot 3.860 \cdot a^2$
For the suspenders, say .... $\left(p_0 + \frac{p_1}{2}\right) \cdot 0.381 \cdot a^2$

Together, $Sl = 7.27 p_0 a^2 + 4.05 p_1 a^2$.

For $p_0 = \frac{p}{3}$ or $p_1 = \frac{2p}{3}$, $\qquad Sl = 5.124 pa^2$;

$p_0 = \frac{p}{4}$ or $p_1 = \frac{3p}{4}$, $\qquad Sl = 4.86 pa^2$.

These values indicate the advantage offered by the smaller number of panels and by the best position of diagonals as compared with the coefficients 5.51 and 5.14 of table (47) of § 27.

From those strain-lengths and the weights of the central arch, and from those of the whole 1700-foot span, we may approximately calculate the strain-length of the latter. For it is nearly correct to assume that the weights of the carrying

parts of these structures stand very nearly in the same ratio as their respective strain-lengths. We have for the

$2a =$	1700-ft. arch.	1002-ft. arch.
Dead load plus live load and part of the tower as arising from the last diagonals of the cantilevers	3.87* tons	3.05 tons
Of which floor plus live load	1.5 "	1.5 "
Wind-bracing	0.357 "	0.2 "
And the carrying-frames	2.013 "	1.35 "

To find c for the strain-length of the great span we now use the equation

$$c \cdot \frac{850^2 \cdot 3.87}{850 \cdot 2.013} = 5.124 \frac{501^2 \cdot 3.05}{501 \cdot 1.35}. \quad c = 3.55.$$

But the strain-length of the whole cantilever-arch can also be calculated directly from the strain-sheet, and doing this there will be found—

For the two cantilevers, $2.06pa^2$; } Together
For the middle arch, $1.40pa^2$. } $3.46pa^2$. †

The two coefficients differing only by 2.5 per cent proves the accuracy of the method used. Of the arch with five hinges the anchorage moment being $0.4 \cdot 0.43pa^2 = 0.174pa^2$, the strain-length of two natural anchorages will be $0.348 \cdot 3.16pa^2 = 1.1pa^2$.

* The strains of the last diagonals of one of the cantilevers have the vertical component of 873 tons; and the weight of one outside tower being 242 tons per track, we obtain $4 \cdot \frac{873}{1932} \cdot 242 = 440$ tons and $\frac{9124 + 0.1 \cdot 1700 + 440}{2 \cdot 1700} = 2.87$ tons, and with live load 3.87 tons.

† Average $3.51pa^2$.

For the ordinary cantilever truss-bridge the coefficient would be 2.72 instead of 1.1.

The total strain-length of the cantilever-arch therefore is

$$Sl = 4.61pa^2.$$

We may now compare three types as follows: the cantilever arch-bridge arranged as on Plate VII.; the ordinary arch with three hinges, twelve panels, and a depth of one seventh; the ordinary cantilever truss-bridge, so arranged that the strain-length between the towers becomes a theoretical minimum. We find for the

	Cantilever-arch	Ordinary arch	Cantilever-truss
between the towers	$3.51p'a^2$	$4.86p''a^2$	$5.7p'''a^2$

Where the p are the greater, the greater are the coefficients, though the p do not simply increase in the same ratio as the coefficients.

The ordinary arch has the coefficient 4.86 because, as will be easily seen, the quotient $\frac{p_0}{p_0 + p_1}$ would be $\frac{1}{4}$ for a span of 1700 feet, the same quotient having been $\frac{1}{3}$ for the 1002-foot arch.

The values of the total strain-lengths of the three types would be found $4.61p'a^2$, $4.86p''a^2$, $8.07p'''a^2$, where again the p grow with the coefficients, though not in the same ratio.

And the strain-length of a 14-panel Whipple truss, with a depth of one seventh of the span, was $7.32pa^2$, where, however, p is greater than p''' of the cantilever truss-bridge.

It is obvious that the cantilever arch-bridge is by far the best of all these types, especially for very great spans.

Applying the formula (39) of § 22, the limiting span (under the specification) for the central arch would be

$$y = 1002\left(1 + \frac{1.70}{1.35}\right) = 2263 \text{ feet}$$

where p_0 is supposed equal to $\frac{p}{3}$ of the 1002-foot span, and the limiting span of the cantilever-arch would become

$$y_1 = 2263 \cdot \frac{5.124}{3.51} = 2263 \cdot 1.46 = 3304 \text{ feet.}$$

But applying formula (39) directly to the cantilever-arch, the limiting span is $1700\left(1 + \frac{3.870}{2.013}\right) = 3267 = y_1'$. The difference of y_1 and y_1' is a little over one per cent.

Though these results can be looked upon as vouchers for the correctness of the estimated weights and of the method used, it must not be believed that y_1 and y_1' are the theoretical limiting spans of the cantilever-arch. It must be remembered that with bowstring-girders, arches, suspension-bridges, etc., the value of the strain-length depends also on the proportion of movable load to permanent load.

Thus, for instance, the middle arch for $p_0 = 0$ has $c = 4.05$; and if also the wind-bracing and the vertical posts were considered as non-existing, for strains of $7\frac{1}{2}$ tons per square inch the limiting span would be found 3320 feet instead of 2263.

Finally, the strain-sheet of the 1700-foot span (Plate VIII.) shows that the tensional forces of a cantilever arch-bridge are comparatively small and the tensional joints can be readily made. This property constitutes a good feature of the system, which thereby becomes well adapted for very great spans.

Of all systems thus far examined it unites the least strain-length and weight between the piers (equal strains being supposed) with the least total strain-length and with the smallest relative tensile strains. Besides, it is no more difficult to erect the cantilever arch-bridge than to erect a cantilever truss-bridge.

§ 32. The Cantilever Arch-Bridge of 360 feet Span across the Magdalena near Honda in New Granada.

Plate IX. represents the Magdalena bridge at Honda. It is designed to carry the rolling-stock of a railroad of 3 feet gauge. On account of the severe grade of the approaches only light trains can pass the bridge. It was therefore considered more important to provide for heavy engines or for concentrated panel-loads than for great general rolling-loads.

The bridge is proportioned to carry two 30-ton tank-engines, with 20 tons on a wheel-base of 7 feet, or with 30 tons on a 15-foot base, followed by the heaviest loaded narrow-gauge freight cars known.

The bridge being protected as regards wind by the mountains between which the river winds its way, a maximum wind-pressure of 750 pounds per lineal foot of bridge was considered sufficiently high.

The floor was specified to be at least 10 feet wide, and accordingly the central distance of top-chords was made 10 feet.

The height from the top of the rails to the lowest points of the arch in the centre is made as small as possible, namely, 4 feet 8 inches.

The wooden floor-beams rest upon the top-chords, and these are correspondingly designed.

The flooring consists of 3-inch planks and of two 6x12-inch rail-stringers laid flat.

The floor is limited by wrought-iron railings made of angle-standards bolted to the side-plates of the top-chords, and of longitudinal angles. The diagonal bars are made of flat-iron.

The frames of the arches lie in inclined planes of which the inclination is 1:4 on each side. The parabolic arch is

52 feet deep. The end-posts are 54 feet high. The width between the centre points of the feet of the end-posts is $10 + \frac{54}{2} = 37$ feet. Notwithstanding this considerable width, the bridge being very light, only a bare stability against wind-pressure is secured, and each main-post and end arch-piece is therefore bolted to the abutment masonry with four $1\frac{1}{2}$-inch bolts with upset ends, 8 feet 6 inches long, which act against anchor-plates of corresponding size.

Each of the twenty-four panels of the bridge is 22 feet 6 inches long. The east approach consists of a 30-foot span. The arch itself occupies sixteen panels, ten of which belong to the independent span.

All lateral and transverse wind-braces are tie-struts, consisting of various sizes of angles up to pairs of the heaviest $6\frac{1}{8}$x$6\frac{1}{8}$-inch angles.

Only the four last diagonal wind-braces of the arch proper at each abutment are swelled tie-struts; all others are single angles or pairs of angles.

There are no further lattice-bars in the bridge. The trough-shaped arches and top-chords are stiffened by diaphragms which give the needed rigidity in order to prevent deformation during transportation.

The details on Plate VIII. show how the central intersection of middle lines was secured for the wind-braces as well as for the main-braces. Each pair of corresponding arch-pieces with two lateral strut-angles and two stiff wind-diagonals forms one complete frame to be lifted bodily by means of a travelling erecting-machine 64 feet long, weighing with crabs, chains, etc., 5400 kilos, and designed to carry 10 tons.

The maximum strain per square inch of any member of the bridge is less than 10,000 pounds.

The tensions of the top-chords during erection are 16,000 pounds.

All iron is of very superior quality. The average of 38 tests of heavy round bars, of angles, and of plates yielded 23.48 tons per square inch (52,600 lbs.), with 35 per cent of contraction of area and 21.7 per cent of elongation, on test-pieces 8 inches long. The testing-machine was of the lever description, well built and in constant use by government inspectors. The plates especially were of superior quality, yielding 22.5 tons, with 16.4 per cent extension *across* the fibres. The $\frac{5}{16}$-inch plates gave still better results. They stood 25 tons lengthwise, with $21\frac{1}{2}$ per cent elongation and with 43 per cent contraction of area.

All channel-bars used for the long posts are 8 inches wide, and were made of steel of 32.4 tons tensile strength, but were treated like iron. No plate of the bridge is less than $\frac{5}{16}$ inch thick.

All holes throughout the bridge were drilled and reamed. A small conicity is given between the bodies and the heads of rivets. In order to facilitate erection, and to be able to fix the posts rigidly to the chords and arches, each joint has two 4-inch pins. The pins by which the arch is connected with the viaduct-chords are 6 inches in diameter. The web-diagonals of the arch are round upset-bars without welds and with forged separate eye-bolts. All diagonals are provided with at least two right and left screw-couplings. No welds were admitted. The bolts needed to splice the long posts and to connect the top-chords of the arch and of the viaduct are turned and tapering; their holes are reamed. All other bolts and the pins have a driving fit.

All parts of the whole structure were assembled. Each separate joint was carefully fitted, and the parts belonging to the joint were countermarked in the most systematic and thorough manner.

The iron was first freed from scale and rust in a trough filled with diluted acid; thereupon it was brushed and was

put into another trough containing boiling water. The iron taken from this bath dried quickly, and while still hot received a coat of oil. The finished work finally had a coat of metallic paint given it.

The arrangement for separation of the top-chords of the middle arch from those of the cantilevers consists of well-fitting wrought-iron washer-blocks, of pairs of 3½-inch bolts, and of pairs of cast-iron sand-boxes. The latter were tested.

The posts of the arch and of the viaduct, being in part of considerable length, were made very strong. The maximum pressure of the main-posts is only 2 tons per square inch, and of the longest trestle-posts it is about 1 ton per sq. inch.

The web-diagonals are strained from 5000 to less than 8000 pounds per square inch.

Before the design was decided upon the author studied the subject most carefully. The fixed continuous beam with its changeable angles of anchorage, also various modifications of cantilever truss-bridges, and for comparison also the Whipple truss, were considered.

It was found that the latter would have weighed 1632 pounds per lineal foot, and that a cantilever truss-bridge, including the iron of its anchorages, would have required 1940 pounds per foot. The actual weights of the arch-bridge were found to be almost ½ per cent lighter than those estimated.

The permanent weights of the cantilever-arch are distributed as follows:

	Metrical Tons.
Frames of the arch	125,871
Stringers (material included in the top-chords)	23,400
Wind-bracing	23,573
Railing, floor-bolts, spikes, pin-sleeves	7,571
Wood and rails	34,280
	214,695
Iron and steel per lineal foot	½ ton.

The strain-length of the strain-sheet is found to be 5.96 pa^2. An ordinary 16-panel arch of 360 feet span with $h = 52$ feet would have given 6.41pa^2.

The qualities of the bridge differ from those of the design exhibited on Plate VII. in this regard, that the depth of the independent span is only one eleventh instead of one seventh of the span. Hence its great strain-length of 8.9pa^2.

Of course a Whipple deck-truss could not have been built at Honda. Nor could it have been only 10 feet wide. For a depth of 52 feet a width of about 24 feet would have been required. A complete system of iron floor-beams and of separate stringers would have been necessary.

The anchorage of the 360-foot arch at Honda, including the anchorage material contained in the viaduct-chords, weighs about 14 tons. And there is still the anchorage masonry to be added. The cost of these two items just about balances the cost of the greater weight of a 360-foot Whipple truss-bridge. But the two iron towers which would have been needed for a Whipple bridge are saved.

The iron-work of the Honda bridge was manufactured in Germany, and the structure is now in process of erection.

§ 33. The Combination of a Loaded Elastic Beam suspended from a Flexible Parabolic Catenary.

In § 26 it was stated that a suspended girder is relieved by the deformation of the catenary. The combination of the elastic beam suspended from a flexible catenary, certain suppositions being agreed upon, admits of a strict analytical treatment, and it has become useful for the very greatest spans, since steel wire of great strength and length can be produced in large masses at reasonable cost.* Coils of wire

*** During the Vienna Exhibition in the year 1873 a coil of No. 18 wire 3½ miles**

of one hundredweight and even two hundredweight in one piece can now be produced. Thus single wires of one hundredweight drawn to No. 7 B. W. gauge will be 1300 feet long; and since the capacity of elastic material for absorbing tensional impacts within the elastic limit increases in direct ratio to the length, the capacity of such long wires to withstand moderate impacts is at least twenty-five times that of steel bars 50 feet long. This advantage is enhanced by the very high elastic limit of steel wire.

The manufacture of wire itself is a guarantee of its uniformity, and wire can be tested very easily. The art of making wire cables for suspension-bridges has been brought to an exceptionally high degree of perfection by the late Johann Roebling, who emigrated from the centre of the old wire-manufacturing province of Germany, and by his son, W. Roebling. The cables of the Niagara, the Cincinnati, or of the New York and Brooklyn suspension-bridge leave hardly anything to be desired. The latter bridge, of nearly 1600 feet span, is a proof of the possibility of erection without danger of still greater spans. The wires of the cables of the New York bridge are connected by right- and left-hand screw-couplings with variable depth of thread, or with so-called "vanishing" thread, so that only about 5 to 10 per cent of the original strength of the wire is lost. The wires are galvanized and varnished; the cables are wrapped all over and well painted. The finished cables

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long weighing 84 pounds, and another of No. 3 wire 1155 feet long weighing 204 pounds, made without any special or patented machinery, were exhibited.

Lately the manufacture of wire has assumed enormous proportions in Germany. Besides satisfying the home market the works exported in the years 1879 to 1883 respectively 70,000, 102.000, 156,000, 227,000, 250.000 tons, or from double to two and a half times the British exports, of which, besides, one half must be considered as German wire imported and exported to other countries. This extension of the manufacture has made wire a very economical product in those countries where the price is not artificially raised.

offer the enormous strength of not less than 75 tons per square inch of metallic area. For the reasons given a lower factor of safety is admissible than for riveted work, and even for iron or steel eyebars. Good iron bars bear an ultimate strain of 50,000 pounds per square inch, and eyebars of steel if *properly* annealed must not be rated above 70,000 pounds. Such bars 25 feet long and 6 inches wide require 16 per cent of additional material for eyes and pins; if the bars are 3 inches wide, only 8 per cent is needed.

The cables or chains and the suspended beam are all that is required to build an economical stiffened suspension-bridge.

Additional members, such as inclined stays, are not only not needed, but, being of the character of redundant members, lead to waste of material and labor. A somewhat intricate analysis will show that such stays, causing a suspended beam to act like a continuous girder with yielding supports, will cause high moments of flexure at their points of attachment to the beam. Besides, the changes of temperature disturb all conditions of intended equilibrium.

It is true that if, according to the author's suggestion, inclined suspenders are attached to their corresponding vertical suspenders by means of levers or pulleys, any fixed relation of the strains of the two suspenders at a point can be introduced.

But also in this case it will be found that greater complexity does not produce greater economy.

§ 34. Theory of Equilibrium of a Loaded Homogeneous Elastic Beam Suspended from a Flexible Catenary.* (Pl. VIII.)

Let there be denoted by

ACB, the originally parabolic catenary.

$A'C'B'$, the cable in equilibrium under the loads.

$2b = AB = A'B' = O'O''$, the span.

H, the original depth of the catenary.

$y = \varphi(x) = h - \frac{2H.x}{b} + \frac{H.x^2}{b^2}$, its original equation.

O', the origin of the co-ordinates.

$O'O''$, the positive side of the abscissæ x.

$O'A$, the positive side of the ordinates y.

π_x, the intensity for the abscissæ x of the forces acting in suspenders, supposed to be close together, and representing a force per unit of length of the x.

Q, the constant horizontal component of the tension of the cable $A'C'B'$ for any x, it being the result of *all* the weights acting on the system.

E, the modulus of elasticity, supposed to be constant throughout.

R_1, a possible positive reaction, the $+$ sign being used for forces acting upwards towards the positive y.

$2b.k$, a uniform permanent weight, k its intensity per unit of length of the beam.

$z.p$, another uniform and movable weight, $z = O'z$ being variable.

* This theory was first worked out by the author in the winter of 1880–1881; it was alluded to in *Engineering* of May 13, 1881, page 487, and in November 1881 it appeared in Van Nostrand's *Engineering Magazine*. The above rendering is an improved and extended one, where the influence of the web on the deflections is considered in accordance with the theory which the author worked out and here publishes for the first time.

M_0', M_0'', the moments caused by p and k for O' and O'' respectively.

These moments can include original end-moments of the beam if it were supposed to be built into the piers. If these were m' and m'', the moment M_0' would include $(m' - m'')$ and M_0'' would include $(m'' - m')$. The moments M_0' and and M_0'' are connected by the relation

$$M_0' + M_0'' = 2b(2b \,.\, k + z \,.\, p) = \text{Span} \times \text{Sum of all the weights.}$$

I, the moment of inertia of the beam $= \dfrac{h'^2}{2}\,\Omega.$

h', d, c, the lengths of the vertical posts, tensile diagonals, and the panels, with w', w'' Ω their areas. All these areas are supposed to be constant mean values so taken that they have the same effect as regards deflections as the real areas which may be varied.

A positive moment is one which causes pressure in the top-chords; it goes with $+\dfrac{d^2y}{dx^2}$.

A negative moment causes pressure below, and goes with

$$-\frac{d^2y}{dx^2}.$$

A shearing force is considered positive if directed in a sense opposite to that of the weights on the bridge. For a certain x it equals $R_{\prime}$ + all forces $\pi_x \,.\, dx$ and $-$ the weights between O' and the point x considered.

We have to find the equation of the elastic curve of the beam

$$\eta = f(x),$$

where η are ordinates as distinguished from those of the cable, which were denoted by y.

If the beam deflects (negative η), and if the small extension of the suspenders which is variable and can be considered in a calculation of correction is neglected, the new ordinate y' of the curve $A'B'C'$ will be

$$y' = \psi(x) = \varphi(x) + f(x). \quad \ldots \ldots \quad (1)$$

This simple equation characterizes the connection between the two elements of the combination.

The equation (1) can be developed in various ways:

(1) By comparing for $x = x_{,}$ the moments, namely, $EI\frac{d^2\eta}{dx^2}$, and the equal value which can be calculated from the loads p, k, and π_x. This leads to the most complex calculations possible in the present case.

(2) By comparing the shearing forces; and

(3) By comparing their increments.

This is the simplest way of obtaining the differential equation needed.

According to Euler's fundamental equation, the moment of flexure of the beam is

$$EI\frac{d^2\eta}{dx^2}.$$

Its differential quotient is the shearing force, and its second differential quotient is the increment of the shearing force. Or there is for the branch Oz',

$$EI\frac{d^4\eta}{dx^4} = \pi_x - p - k; \quad \ldots \ldots \quad (2)$$

or for the branch zO'',

$$EI\frac{d^4\eta}{dx^4} = \pi_x - k. \quad \ldots \ldots \quad (2a)$$

We have now to find the value of π_x.

The first differential quotient of the equation of the new catenary multiplied by Q is the vertical component of the strain of the cable.

Hence Q multiplied by the *second* differential quotient of $\psi(x)$ is simply π_x, which is the increment of the vertical component of the new cable-strain. Thus we have

$$\pi_x = Q \cdot \frac{d^2\psi(x)}{dx^2} = Q \cdot \left[\frac{d^2\varphi(x)}{dx^2} + \frac{d^2f(x)}{dx^2}\right];$$

$$\pi_x = Q \cdot \left[\frac{2H}{b^2} + \frac{d^2\eta}{dx^2}\right].$$

This value introduced into (2) brings

$$EI\frac{d^4\eta}{dx^4}Q = \frac{d^2\eta}{dx^2} + \left[\frac{2H}{b^2} \cdot Q - p - k\right]. \quad . \quad . \quad (3)$$

Before integrating this equation we remark that it is not difficult to consider the unequal length of the suspenders. It is only necessary to multiply Q by a factor $\left(1 + \frac{2H}{rb^2}\right)$, where $r = \frac{sE}{Q}$, s being the section of suspenders per unit of length of the bridge.

We introduce $EI = t^2 . Q$.

$$z = \frac{d^2\eta}{dx^2} + \left[\frac{2H}{b^2} - \frac{p+k}{Q}\right]; \qquad \text{or} \qquad t^2\frac{d^2z}{dx^2} = z.$$

$$z = A \,.\, e^{\frac{x}{t}} + Be^{-\frac{x}{t}}$$

where $e = 2.71828$. Hence we get

$$\frac{d^2\eta_1}{dx^2} = Ae^{\frac{x}{t}} + Be^{-\frac{x}{t}} + \left[\frac{p+k}{Q} - \frac{2H}{b^2}\right];$$

$$f_{\prime}(x) = \eta_{\prime} = t^2 Ae^{\frac{x}{t}} + t^2 . Be^{-\frac{x}{t}} - F_{\prime}\frac{x^2}{2} + Cx + D. \quad (4)$$

where
$$-F_{\prime} = \left(\frac{p+k}{Q} - \frac{2H}{b^2}\right).$$

Denoting also by

$$-F_{\prime\prime} = \left(\frac{k}{Q} - \frac{2H}{b^2}\right),$$

and keeping the same origin O' of the co-ordinates, we obtain for the branch zO'',

$$f_{\prime\prime}(x) = \eta_{\prime\prime} = t^2 \alpha e^{\frac{2b-x}{t}} + t^2 . \beta . e^{-\frac{2b-x}{t}} - F_{\prime\prime} . \frac{(2b-x)^2}{2}$$

$$+ \gamma(2b - x) + \delta. \quad . \quad . \quad . \quad . \quad (4a)$$

Before the constants of integration are determined, it will be well to remember that thus far the formulæ (2) to (4) only give the flexures as caused by the chords, so that the deflections are just sufficient to make the exterior work of the forces p, k, and π_x equal to the work of the chords. And as we know that the web of a well-designed truss equals about that of the chords, the deflections must be much greater. For a truss fully loaded they should on the average be *double* as great as those found according to the theory of Euler, which holds approximately good for a full beam with very heavy web. This consideration shows also plainly that the *usual theory of*

continuous girders as applied to open trusses is utterly false, and indeed quite worthless unless the web is considered.

If
$$r^2 = \frac{1}{2c}\left(\frac{\Omega}{w'} h'^2 + \frac{\Omega}{w''} d^2\right),$$

the true equation of the elastic line of a truss with a very great number of posts and systems is

$$EI\frac{d^2\eta}{dx^2} = M_x - r^2 . \frac{d^2M_x}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation twice differentiated gives

$$EI\frac{d^4\eta}{dx^4} = \frac{d^2M_x}{dx^2} - r^2\frac{d^4M_x}{dx^4}.$$

But as
$$\frac{d^2M_x}{dx^2} = \pi_x - p - k,$$

there follows
$$\frac{d^4M_x}{dx^4} = \frac{d^2\pi_x}{dx^2};$$

and as
$$\pi_x = Q\left[\frac{2H}{b^2} + \frac{d^2\eta}{dx^2}\right],$$

there is
$$\frac{d^2\pi_x}{dx^2} = Q . \frac{d^4\eta}{dx^4} = \frac{d^4M_x}{dx^4};$$

and finally,
$$\frac{EI + r^2Q}{Q} . \frac{d^4\eta}{dx^4} = \frac{d^2\eta}{dx^2} + F_r. \quad . \quad . \quad . \quad . \quad (3a)$$

If now, instead of t^2, we put

$$\frac{EI}{Q} + r^2 = t'^2, \qquad \text{or} \qquad t'^2 = r^2 + t^2,$$

the equation (3*a*) corresponds exactly with (3), except that another I is substituted, namely,

$$I' = I + r^2 . Q.$$

Thus all the formulæ which are derived in this exposition from Euler's equation remain correct, and, what is the essential point, the deflections as caused by the web are considered. The interior work will now be found to agree with the exterior, and the axiom that action and reaction must be equal is no longer violated.

Indeed equation (5) if *properly* applied to continuous girders* also removes the particular blame to which we have alluded. It can be divided in two equations. Web and chords can be treated separately, and the effects can be added where—as with simple girders or continuous girders—the principle of superposition of effects holds good. This principle must *not* be confounded with the superposition of *different systems*, such as the combination of catenary and beam.

Now we shall determine the constants.

We must first consider those conditions which apply to all combinations of beam and catenary, no matter how the *end conditions, compatible with those very conditions*, are selected.

(1) The algebraic sum of weights equals the sum of vertical reactions. The algebraic sum of moments of exterior forces—including reactions, and m' and m'', if any—must vanish for any fulcrum.

These two conditions are also fulfilled if the latter one is applied to O' and O''.

Taking the fulcrum O'', the moment of the weights (including $m'' - m'$, if existing) is M_0''.

At A' the reaction of the pillar is $- Q . \psi'(0)$.

* But it will be found that the moments over the piers for the suppositions made become so small that this kind of continuity may be considered as practically useless.

The reaction of the beam at O' is

$$R_{,} = EI\frac{d^3\eta_0}{dx^3} \quad \text{or} \quad M_0'' = 2b\left[R_{,} - Q.\frac{d\psi(0)}{dx}\right].$$

If we differentiate (4) three times, multiply by EI, and finally put $x = 0$, we obtain $R_{,}$.

If we differentiate $\psi(x) = \varphi(x) + f(x)$ once, multiply by $-Q$, and finally put $x = 0$, we obtain the reaction at A'. Thus we obtain

$$\left.\begin{aligned} C &= \frac{2H}{b} - \frac{M_0''}{2b.Q}; \\ \text{and in a similar manner,} \\ \gamma &= \frac{2H}{b} - \frac{M_0'}{2b.Q}. \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

(2) The two branches $\eta_{,}$ and $\eta_{,,}$ must be identical for $x = z$. This condition requires that for $x = z$,

$$\eta_{,} = \eta_{,,}; \qquad d\eta_{,} = d\eta_{,,}; \qquad d^2\eta_{,} = d^2\eta_{,,}; \qquad d^3\eta_{,} = d^3\eta_{,,};$$

or that the deflections, the tangential angles, the moments of flexure, and the shearing forces must be equal for $x = z$.

These conditions lead to the following equations:

$$\left.\begin{aligned} &(a)\ t^2Ae^{\frac{z}{t}} + t^2Be^{-\frac{z}{t}} - F_{,}\frac{z^2}{2} + Cz + D = t^2\alpha e^{\frac{2b-z}{t}} \\ &\qquad + t^2\beta e^{-\frac{2b-z}{t}} - F_{,,}\frac{(2b-z)^2}{2} + \gamma(2b-z) + \delta; \\ &(b)\ tAe^{\frac{z}{t}} - tBe^{-\frac{z}{t}} - F_{,}z + C = -t\alpha e^{\frac{2b-z}{t}} \\ &\qquad + t\beta e^{-\frac{2b-z}{t}} + F_{,,}.(2b-z) - \gamma; \\ &(c)\ Ae^{\frac{z}{t}} + Be^{-\frac{z}{t}} - F_{,} = \alpha e^{\frac{2b-z}{t}} + \beta.e^{-\frac{2b-z}{t}} - F_{,,}; \\ &(d)\ A.e^{\frac{z}{t}} - Be^{-\frac{z}{t}} = -\alpha e^{\frac{2b-z}{t}} + \beta.e^{-\frac{2b-z}{t}}. \end{aligned}\right\} \quad (7)$$

From (a) and (c) there results

$$D - \delta = p \cdot \frac{t^2}{Q} \quad \ldots\ldots \quad (8)$$

The equations b, c, d are really only two, and these give

$$\left.\begin{aligned} A &= \beta \,.\, e^{-\frac{2b}{t}} - \frac{p}{2Q}\, e^{-\frac{z}{t}}; \\ B &= \alpha \,.\, e^{\frac{2b}{t}} - \frac{p}{2Q} \cdot e^{\frac{z}{t}}. \end{aligned}\right\} \quad \ldots\ldots \quad (9)$$

(3) The curve of deflection of the beam must agree with the curve of the cable which has stretched, also with the extensions of the back-cables and anchorage-chains, with the deflections caused by the compression of masonry, with the deflections due to the expansions and contractions as caused by temperature, the extensions of the suspenders and the compressions of the pillars. The area w of the cable must be determined, so that for the maximum Q its strain per square unit becomes a fixed quantity. The degree of compression of masonry is unknown and cannot be considered. The equation of the work of the beam is already satisfied by equation (5). There remains only the cable to be considered.

The specific cable-strain of the part $A'C'B'$ is

$$\frac{Q}{w} \cdot \frac{ds}{dx}.$$

Its extension Δds for the length of the element ds is

$$\frac{Q}{w \,.\, E} \cdot \left(\frac{ds}{dx}\right) . \, ds$$

where the last ds is the original length of the cable between the imaginary sections at x and at $(x + dx)$.

The element of work becomes

$$\frac{1}{2}\frac{Q}{w\,.\,E}\frac{ds}{dx}\,.\,ds\,.\,Q\,.\,\frac{ds}{dx}=\frac{Q^2}{2wE}\left(\frac{ds}{dx}\right)^2.\,ds,$$

and the whole work from A' to B' is

$$\frac{Q^2}{2E}\int_0^{2b}\frac{ds}{w}\left(\frac{ds}{dx}\right)^2.$$

In a similar manner we find for the work done by the back-cables and anchorages

$$\frac{Q^2}{2E}\cdot\frac{W}{w},$$

where W is a constant.

For $\pm\, t^\circ$ of changes of temperature the length ds changes by

$$\pm\,\frac{t}{f}\,.\,ds.$$

The change of temperature takes place from that temperature at which the bridge was adjusted after erection, or after the Q from permanent loads was produced. The force $Q\,.\,\frac{ds}{dx}$ which must be multiplied by the change of length $\pm\,\frac{t}{f}\,.\,ds$ should be the mean between the original $Q\,\frac{ds}{dx}$ and that produced by movable and permanent loads together.

Considering, however, the uncertainty of the temperature to be assumed, the approximate value

$$\pm\,\frac{t}{2f}\cdot\frac{p+2k}{2p+2k}\cdot Q\,.\int\frac{ds^2}{dx}=\pm\,\tfrac{1}{2}Q\,.\,T\int\frac{ds^2}{dx}$$

may be taken.

Finally, we may safely assume that only wire cables will be used, so that the area w is a constant.

We obtain for the cable

$$\frac{Q^2}{Ew}\left[W + \int_0^{2b}\left(\frac{ds}{dx}\right)^2 ds\right] \pm QT\int \frac{ds^2}{dx}$$
$$= \left[A_{,} . \eta_0 + B_{,} . \eta_{2b} - \int_0^z \pi_x . f_{,}(x) . dx - \int_z^{2b} \pi_x . f_{,,}(x) dx.\right] \tag{10}$$

This equation is very complex, but also very instructive. It shows that w appears only once in all equations, and that $\frac{Q}{Ew}$ is a measure of the flexibility of the bridge. The greater Q as caused by *permanent* loads as well as by movable ones, and as caused by *flat catenaries*, the greater the work done by the cable, the less the work to be done by the beam. The smaller E the less the work for the beam. If, for instance, we could use India-rubber instead of steel, the curve $\psi(x)$ would deviate *much* from the original parabola; it would be near to the imaginary funicular line for which a beam would no longer be needed, provided such great deformations were admissible.

The better the material, the smaller the area w can be taken, the lighter the beam will become.

The rope-dancer, to give an illustration, uses the very best material for his purpose, strains it artificially with a great permanent Q, and uses a very flat catenary. The better the material of his rope, the greater Q, the smaller will be the necessary deflections, the greater will be his safety.

It is for this reason, in addition to the saving of weight, that steel-wire cables with working strains of 18 or 20 tons per square inch will be so useful for large suspension-bridges.

Equation (10) also shows the influence of temperature. It obeys the same intricate law expressed by the complex

formula (10). The temperature can no more be considered without permanent or movable load than it is possible to consider the movable load without the permanent one. We have to do with a superposition of two different systems. The *distribution* of the strains over these two systems is no longer of such a nature that the mere law of superposition of effects holds good.

The influence of changing temperatures is very considerable with suspension-bridges because of the long back-cables. For a 1600-foot span $H = 125$ feet, and $\pm 60°$ Fahr.; the total cable exposed to change of length may be taken to be 2900 feet long. Hence half of $A'C'B'$ would be longer or shorter by 0.58 of a foot.

The natural deflection or rise of the centre of the cable would be $\pm 2' 9''$, or together $\frac{1}{300}$ of the span.

Considering that railway truss-bridges by their movable loads deflect to the amount of $\frac{1}{1000}$ to $\frac{1}{1500}$, the above approximate calculation shows how injudicious and how wasteful it would be to make the beam *continuous*, because it would become very shallow in order to deflect so much, or else the bridge would be no longer a suspension-bridge, but a truss-bridge assisted by a cable.

The equations (5), (8), (9), (10) give six conditions for the nine unknown coefficients (A, B, C, D, α β, γ, δ, and Q). Therefore only three other arbitrary conditions can be imposed as regards the ends of the beam.

This important conclusion excludes at once the notion of fixing a beam at both ends so as to produce moments m' and m'' changeable with the loads. If the beam were anchored to the piers, the anchorage would do a certain amount of work with, say, a strain-length of $6M$.

This means work, and work means deflections.

The beam would have a changeable angle $\frac{d\eta}{dx}$ at each pier,

and these angles would be equal $n \,.\, m'$ or $n \,.\, m''$, for full load about $\frac{1}{400}$, which in a cantilever of 800 feet would cause a deflection of 2 feet.

The idea of building the beam into the masonry also implies that either

$$\eta_0 = \eta_{2b} = 0,$$

or else that there are springs to carry the ends of the beam.

In this case

$$\eta_0 = g \,.\, R_1 \qquad \text{and} \qquad \eta_{2b} = g \,.\, R_2.$$

Hence the idea of a beam fixed into the piers implies four new equations, which are of such a nature that the three equations (8) and (9) can *not* be reduced to two.

If, nevertheless, the idea would be carried out, a changeable number of suspenders would cease to act for each mode of loading; that is, the whole calculation would lead to nonsense.

The end-deflections η_0 and η_{2b} must be provided for, because the suspenders extend and the piers become shorter by compression. Besides, a continuous beam would be more costly than one with Koepke's central hinge; it would not be stiffer, but it would give greater deflections than the latter. This is quite obvious, because it has to do more work.

The cable determines the vertical stiffness—that is, the average deflections—more than the beam.

A fixed continuous beam would be quite visionary, for the points of reversion of flexure would be very near to the piers, and the anchorage would involve great expense as regards metal and masonry in addition to reduced stiffness and greater strains, greater cost between the piers, not to mention the uncertainty of the actual strains owing to the deflections due to compression of anchorage-masonry at four different points.

For all these good reasons it is obvious that if in future great suspension-bridges are to be built, they will have beams hinged in the middle. This is the only practical arrangement possible.

In order to accommodate the changing lengths of the suspenders, and to secure their working according to theory, either a little play must be given at the ends, or springs must be introduced, or else a number of suspenders must be removed. We reserve these considerations for a calculation of correction.

The hinge does away with extra strains as caused by temperature, etc., on the continuous beam. It renders us independent of equation (10) for the first calculation. It results that—save some slight secondary strains—the equations (6) and (9) and the two end conditions $m' = m'' = 0$ of the hinged beam with $M_b = 0$ are quite sufficient to calculate all those constants which interest us mainly. The constants D and δ interest us only as regards the calculation of corrections. The condition of the hinge makes us independent of w, which we may assume as we please. For full load there will be no primary strains in the beam, whereas the continuous beam would have heavy strains under full load.

We make now

$$m' = m'' = 0,$$

or
$$A + B = F_{,}; \qquad \alpha + \beta = F_{,,}. \quad . \quad . \quad . \quad . \quad (11)$$

Further, $M_b = 0$; or if $z =$ or $< b$,

$$\alpha . e^{\frac{b}{t}} + \beta e^{-\frac{b}{t}} = F_{,,}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and we obtain

$$2\left(1 - e^{\frac{b}{t}}\right)\left(1 - e^{-\frac{b}{t}}\right) F_{,,} Q = p . \left(1 - e^{\frac{z}{t}}\right)\left(1 - e^{-\frac{z}{t}}\right). \quad (13)$$

For $z = 0$ there is $Q_0 = k \dfrac{b^2}{2H}$;

For $z = b$ there is $Q_b = \left(k + \frac{p}{2}\right)\frac{b^2}{2H}$;

For $z = 2b$ there is $Q_{2b} = (k + p)\frac{b^2}{2H}$.

For intermediate values we assume Q, and calculate the corresponding z. Denoting by

$$\left.\begin{array}{l} \frac{u}{t} \quad \text{the value} \quad \left[1 + \left(\frac{e^{\frac{b}{t}} + e^{-\frac{b}{t}}}{2} - 1\right)\frac{b}{2F_{\prime\prime}Q}\right], \\ \text{there is found} \quad z = t \log \text{nat}\left(\frac{u}{t} \pm \sqrt{\frac{u^2}{t^2} - 1}\right). \end{array}\right\} \quad (14)$$

Further, there is

$$\left.\begin{array}{l} \beta = \alpha e^{\frac{b}{t}}; \quad \alpha = \frac{F_{\prime\prime}}{1 + e^{\frac{b}{t}}}; \quad \beta = \frac{F_{\prime\prime} . e^{\frac{b}{t}}}{1 + e^{\frac{b}{t}}}; \\ A = \alpha e^{-\frac{b}{t}} - \frac{p}{2Q} . e^{-\frac{z}{t}}; \quad B = \alpha . e^{\frac{2b}{t}} - \frac{p}{2Q} . e^{\frac{z}{t}}. \end{array}\right\} \quad (15)$$

All formulæ developed contain the *assumed* value of I or I', which in fact are just the values to be found.

Differing from the old common theory of stiffened suspension-bridges, these depend, like the moments of flexure, on the permanent loads.

But several examples for varying proportions of $\frac{p_0}{p_1}$ and of $\frac{h}{2a}$ having been calculated, certain values of I will be found which will be smaller than those calculated according to the old theory. In case of a special project, those ratios of co-

ordinate I can be used towards assuming a first approximate value.

The higher are the admissible strains, the greater will be the deformations of the cables, and the greater, therefore, the reduction of the necessary moment of inertia.

It may be desired to build a roadway bridge which rarely, if ever, will be strained to the limit of the specification. It is here that the principle of the *stable* equilibrium of the catenary which has saved so many a weakly designed suspension-bridge becomes useful.

For the limit of flexibility is indicated, if not already by the chord-material required on account of heavy winds, by the greatest admissible *grade* of the floor at O' and O''.

Roadway bridges will be empty at a time when such hurricanes blow for which the lateral stability is calculated. There will be no vertical moments from movable loads during such events.

The grade at O' is

$$\alpha_{,} = t(A - B) + C;$$

and at O'', $$\alpha_{,,} = t(-\alpha + \beta) - \gamma.$$

These equations lead to tolerably manageable formulæ if $\alpha_{,}$ is chosen for the case of a half-loaded bridge.

An initial or positive grade at O' being arranged, a great $\alpha_{,}$ can be admitted and chosen, provided that if the whole bridge be loaded during the hottest day there will be no visible deflection.

To calculate this *camber* the following formulæ may be found useful:

The length of the half-parabola is known to be

$$S_0^a = a\left[1 + \frac{2}{3}\left(\frac{h}{a}\right)^2 - \frac{2}{5}\left(\frac{h}{a}\right)^4 + \frac{4}{7}\left(\frac{h}{a}\right)^6 - \frac{10}{9}\left(\frac{h}{a}\right)^8 + \dots\right];$$

and for small variations of $h = \Delta h$,

$$\Delta S_0^a = \Delta h\left[\frac{4}{3}\frac{h}{a} - \frac{8}{5}\left(\frac{h}{a}\right)^3 + \frac{24}{7}\left(\frac{h}{a}\right)^5 - \frac{80}{9}\left(\frac{h}{a}\right)^7 + \ldots\right].$$

This series can be reversed by introducing

$$\varepsilon = \frac{S}{a} - 1 \qquad \text{and} \qquad \Delta\varepsilon = \frac{\Delta S}{a};$$

whereupon there will be obtained

$$\left(\frac{h}{a}\right)^2 = \frac{3}{2}\varepsilon + \frac{27}{20}\varepsilon^2 - \frac{81}{175}.\varepsilon^3 \ldots;$$

$$\frac{\Delta h}{a} = \Delta\varepsilon . \frac{a}{h}\left(\frac{3}{4} + \frac{27}{20}\varepsilon - \frac{243}{350}\varepsilon^2 \ldots\right).$$

§ 35. The Economic Features of the Suspension-Bridge with Suspended Girders.

The analysis submitted proves the possibility of a more scientific design of suspension-bridges of *any desired degree* of stiffness.

Partially stiffened suspension-bridges may become useful in certain localities and for certain purposes.

Favorable localities are those where the spans must be of extraordinarily great lengths, while the banks of the river are steep, and where the floor is on nearly the same level as the ground, so that anchorage and towers will not become very extensive.

If in addition natural anchorage were in existence, flat catenaries could be adopted, and thus the material needed for the stiffening girders, for the towers, and for the anchorage could be reduced.

The degree of economy of the new system over the old theory, which considers girders to be absolutely stiff, depends on the permanent load, on the rigidity of the girders, and on the sinus versus of the catenary.

It is therefore not possible to express the strain-length by a simple formula, and recourse should be had to examples.

We confine ourselves to giving an idea of the reduction of the shearing forces and moments of flexure by reason of the deformation of the parabolic form of the catenary.

Given: span $2a = 1600$ feet; the moment of inertia of the girders $I = 180{,}000$ square inches $\times$ square feet; the depth of the catenary $= 125$ feet, or $\frac{1}{8}$ of the span; the modulus of elasticity $= 12{,}000$ tons; the permanent load of the bridge $= 3$ tons $= k$ per lineal foot; the movable load $= 1$ ton $= p$; the depth of girders $= 40$ feet.

For the half-loaded bridge

$$z = b = 800 \text{ feet};$$

$$Q = \frac{7}{2} \cdot \frac{800 \cdot 800}{2 \cdot 125} = 8960 \text{ tons};$$

$$t = 491 \text{ feet}; \qquad C_{,,} = -C_{,} = \frac{1}{17920};$$

$$A = -\alpha = -\frac{9.2}{10^6}; \qquad B = -\beta = -\frac{47.14}{10^6}.$$

The moment in the centres of the half-girders, or for $x = \frac{a}{2}$ and $x = \frac{3a}{2}$, is found $\pm$ 30,800 foot-tons, at the rate of $\frac{p \cdot b^2}{20.8}$.

The reactions $\pm$ 166 tons for the points O' and O'' are at the rate of $0.208pa$.

If the bridge had been calculated like a reversed hinged

arch, the stiffness due to the change of form of the cables would have been neglected.

We obtain a comparative table as follows:

	Common theory.	Corrected theory.
Reactions	$\pm$ 200 tons	$\pm$ 166
Advantage by corrected theory		17 per cent.
Moments for $x = \frac{a}{2}$	$\pm$ 40,000 ft.-tons	$\pm$ 30,800
Advantage by corrected theory		23 per cent.
Strains in the chords, tension plus pressure, tons per square inch	8.89	6.85

On these figures was founded the strain-length of a suspension-bridge given at the end of § 26.

§ 36. On the Lateral and Transverse Stiffness of Bridges.

It remains to mention the application of the principles developed in the preceding sections to the lateral and transverse bracing of bridges. This bracing is necessary to bind the compression-members of the panels together into long rigid compound struts, to create *fixed* end-points of the web-struts, to procure floors rigid laterally as well as vertically, to preserve the angles which the web-planes form with the plane of the floor or floors, to reduce the oscillations caused by moving loads, and also to create a reasonable degree of security against impacts.

This security is the more necessary because the connections of the lateral members as well as of the principal members would suffer severely by collisions, even if they were not struck directly.

But the principal lateral strains, or at least those which

can be estimated with some reasonable degree of accuracy, are those caused by high winds.

The task of the bridge-builder should not be considered ended when he has given the necessary vertical strength and stiffness to a structure. On the contrary, his problem is then only half solved. A bridge of which the vertical trusses are simplified at the expense of the scientific attachment of the floor, or by introducing eccentric and loose lateral and oblique or transverse bracing, is a more or less dangerous piece of work.

It is at the points of lateral connections where strains arising from the rigid connections of the posts with the floor-bearers, and secondary strains arising from eccentric attachments of wind-members, are met with. These may cause moments of torsion as well as of flexure. Besides, the wind-diagonals participate in the annihilation of chord-strains, and their assistance may be considered in spans of some considerable length.

The strains arising from wind influence the width between the vertical trusses, and thereby also the extent and cost of masonry. In ordinary truss-bridges the chord-strains should never be reversed. There must always remain tension in one of the bottom-chords, and the other must not be strained so much that the factor of safety falls below 3. The same principle applied to the top-chords, or more generally to the compression-members of any bridge, is more important still.

As regards the maximum wind-pressure to be specified, it may be remarked that, though the frequency of hurricanes depends on the nature of the country in which a structure is located, nevertheless storms of the greatest severity in the course of time may be expected anywhere; and unless a structure is specially protected, either by mountains or by buildings, or because its axis runs in the direction of the

most dangerous storms, it must be designed to meet the strongest hurricanes.

The actual maximum pressure on a flat surface of a square foot area due to wind of 110 miles per hour, according to Didion's experiments on plates of one square metre area, and therefrom reduced to the size of one square foot, is about 33 pounds. The records of experiments on resistance of air agree that the pressure per square foot increases with the area of the resisting surface, and D'Aubuisson has given a formula for size according to which the pressures increase with the coefficient $s^{0.1}$ where s is the surface. Concave spherical surfaces experience double and convex surfaces down to one half and even one quarter of the test-pressure. In calculating the strains arising from wind, the most trying direction of the wind must be supposed, and the whole wind-pressure on both trusses, on the floor, and on the wind-braces must be calculated. Railway bridges must be supposed to be filled with empty box-cars.

The instruments which hitherto have been used to measure wind-pressure usually consist of a receiving test-plate, a wind-vane, an apparatus of transmission of the pressure, and a spring which in a point receives the pressure and by its deflection or compression measures the force.

It is nearly an impossibility to give a trustworthy theory of such an instrument. And the instruments used hitherto being altogether unscientific or unscientifically used, it is not to be wondered at that less is known about wind-pressures than about the resistance of air. The deflections of the spring which meteorologists have used to measure the actual maximum wind-pressure are especially very misleading. It is not by contrivances such as new apparatus for registering or recording the wind, but by some radical improvement in the instrument itself, and in its theory, that our knowledge as regards wind-pressures may be increased.

Though this subject is not directly connected with the principles of economy of design, yet it has an important bearing on the *strains* and on the choice of the dimensions of the wind-bracing of a structure. For this reason we shall now endeavor to give an outline of the theory of the usual spring instrument.

We suppose that the mass of thc plate of the apparatus of transmission of pressure and the reduced mass of the spring itself to be concentrated in the point where the pressure attacks the spring; we further suppose the spring to be without mass, that it is of simple form and not strained beyond the limit of elasticity. We also do not consider friction. The spring at the commencement of the time t is supposed to be without pressure, and without interior strain or velocity.

Let the spring be a blade a long, b broad, d thick, fixed at one end and acted upon by the wind-pressure W at the other end.

Statically a force P produces a deflection y, so that

$$y = P \cdot \frac{a^3}{3 \cdot E \cdot I} = P \cdot \frac{4a^3}{E \cdot b \cdot d^3};$$

or

$$P = y \cdot c, \quad c = \frac{E \cdot b \cdot d^3}{4 \cdot a^3}.$$

Let m be the reduced mass of the spring, etc.; $W = k \cdot f^2$ be the force of the wind, where f is the velocity of the wind.

The momentary force W is annihilated not only by the pressure P corresponding with the deflection y which exists in the same moment of time, but also by the momentary force of acceleration of the mass m, or there is

$$W = c \cdot y + m \cdot \frac{d^2y}{dt^2}.$$

Let us now consider three different laws according to which W is obtained.

(1) Let W act with full force suddenly, which gives

$$y = B \cos rt + \frac{W}{c}; \quad \left(t = 0, \quad y = 0, \quad \frac{dy}{dt} = 0.\right)$$

$$r = \sqrt{\frac{c}{m}}; \quad B = -\frac{W}{c}, \quad \text{or} \quad y = \frac{W}{c}(1 - \cos rt).$$

For

$$r = \sqrt{\frac{c}{m}} = \frac{\pi}{t_{,}}, \quad \text{or} \quad t_{,} = \frac{\pi}{r} = \pi . \sqrt{\frac{m}{c}} = \pi . \sqrt{\frac{4\, ma^3}{d^3 . b . E}},$$

there is
$$y = \frac{W}{c}\left(1 - \cos \frac{\pi t}{t_{,}}\right);$$

and if the time t has reached $t_{,}$ there is

$$y = 2 . \frac{W}{c}.$$

This maximum being reached, and no further action of wind taking place, the spring recoils and undulates about the medium position $\frac{W}{c}$, which is the position of equilibrium of the spring under the constant pressure W.

(2) Let W act gradually, so that for a certain time t the wind-pressure is $W . \frac{t}{t_{,}}$, and that for $t = t_{,}$ the maximum pressure is reached, after which no further action of wind is supposed to take place.

The original equation now becomes

$$m \frac{d^2y}{dt^2} = W . \frac{t}{t_{,}} - cy, \quad \text{and} \quad y = \frac{W . t}{ct_{,}} - \frac{W}{crt_{,}} \sin \frac{\pi t}{t_{,}}.$$

For $t = t_{,}$ the deflection reaches the statical value $\frac{W}{c}$. Thereupon the wind-pressure ceasing, but the mass having the velocity $\frac{2W}{c.t_{,}}$, the deflection increases to nearly $1.2\,\frac{W}{c}$. If the wind-pressure had contiñued with the constant value W, the maximum deflection would have been

$$\frac{W}{c}\left(1 + \frac{2}{\pi}\right) = 1.63\,\frac{W}{c}.$$

(3) Since it is contrary to the laws of nature that the value W can be applied suddenly, let us now suppose that a wind-wave with the velocity $f.\sin\frac{\pi t}{2.t_{,,}}$, or with the pressure $kf^2.\sin^2\frac{\pi t}{2.t_{,,}} = W.\sin^2\frac{\pi t}{2.t_{,,}}$, acts upon the instrument. The equation obtains now

$$m.\frac{d^2y}{dt^2} + c.y = W.\sin^2\frac{\pi.t}{2.t_{,,}},$$

which has the integral

$$y = \frac{W}{2c}\left(1 - \frac{t_{,}^2}{t_{,}^2 - t_{,,}^2}.\cos\frac{\pi t}{t_{,}} + \frac{t_{,,}^2}{t_{,}^2 - t_{,,}^2}\cos\frac{\pi t}{t_{,,}}\right).$$

For $t = t_{,}$ and $t_{,} = \frac{7}{4}\,t_{,,}$ there is found

$$y = 1.42\,.\frac{W}{c};$$

but for $t = t_{,} = t_{,,}$ there is found

$$y = \frac{W}{c}.$$

Or the same wind-wave may produce either the minimum deflection $\frac{W}{c}$ or a deflection 42 per cent greater. The springs of two wind-instruments may be alike, the wind-plates of the same size, etc., but the masses m being different, the times $t_{\prime}$ change, and the same wind-wave produces two essentially different deflections.

Indeed, the differential equation teaches that not only y but also the value $m \cdot \frac{d^2y}{dt^2}$ must be considered.

The usual pressure instruments for wind graphically record the deflections, but the drums move so slowly that the diagrams are not curves but mere zigzag lines.

If the drums had comparatively much greater velocities, or if the mass of the instrument were greater so that the time $t_{\prime} = \pi \cdot \sqrt{\frac{m}{c}}$ were greater, the distorted diagrams could be used to find the points where $\frac{d^2y}{dt^2}$ is zero. At such points the radius of curvature is infinitely great, and the curvature changes from the convex into the concave form. At such points—supposing we have not to deal with previous undulations—the deflection y accurately measures the momentary wind-pressure.

For greater deflections the curves are concave towards the middle line of the diagram, so that $m \frac{d^2y}{dt^2}$ is negative and has to be subtracted; and conversely, if the deflections are less, $m \frac{d^2y}{dt^2}$ has to be added.

In other words, the greatest deflections of the usual diagrams which hitherto by meteorologists were given out to represent the wind-pressure are too great; they would give

wind-pressures which are 42 per cent too great if the law of the wind-pressure of our third supposition were ruling.

Add to this observation that in reality there is not only one wind-wave which has to be considered, but that the indicated deflections are the results of superpositions of vibrations, the difficulty becomes greater still and there are at least two corrections to be made.

The results of our exposition have already removed the discrepancy between the results of experiments with whirling machines (especially by Didion, and in 1879 by St. Loup, in France) and other apparatus, on the resistance of air, and of the maximum wind-pressures said to have been recorded.

At the site of the intended Forth bridge pressures of 65 pounds were recorded by a usual pressure instrument, and over 35 pounds on a fixed board of 300 square feet were found: but this board was struck by the wind at an angle. According to the above exposition, these experiments, though repeating what was already known, cannot be considered as furnishing material of more scientific value than that we already possessed.

Judging from the formula $y = \alpha . \frac{W}{c}$, where α is a certain coefficient, it is clear that if wind acts on a bridge, the coefficient α, among other factors of influence, depends on the mass, also the ballast, of the bridge. Its coefficient c depends on the horizontal statical stiffness of the bridge.

It is a matter of importance in which proportion the time of oscillation of a bridge $(t_{,})$ stands to the duration $(t_{,,})$ of the wind-wave. The time $t_{,}$ depends upon m and c, which therefore together influence the coefficient α. This coefficient α need not be the same as the α of the wind-instrument. There is a field for scientific engineers both to apply their acquaintance with mathematical physics and to endeavor to find the

laws governing the undulations of our statical structures, and more especially the greater strains thereby caused.

Finally, it may be remarked that there is no good reason why, from motives of economy, the wind-pressure should be estimated too low. The question only amounts to the design of substantial web-members, the chords for the lateral and transverse bracing being already given. We have learned from the examples of the preceding sections that the weight of the wind-braces amounts only to a comparatively small part of the whole weight of a bridge, so that parsimony in this regard is out of place.

Economy must be secured by application of the principles of economy, and not by lowering the specification of a bridge.

APPENDIX.

I.

Experiments on Steel and Iron Bridge-Girders.

From the *Railroad Gazette*, Jan. 26, 1883.

As much as sixteen years ago the Dutch Government used steel for parts of its great bridges. But not earlier than 1878 and 1879, on occasion of the construction of the Nymwegen bridge, the Dutch railway-engineers ordered a complete series of experiments to be made, in order to test the real strength of riveted steel girders as compared with those of iron.

Continental government specifications generally prescribe that a certain percentage of finished girders and cross-bearers shall be tested, and therefore a good opportunity was offered to thoroughly test—without great extra cost—with the aid of scientifically constructed apparatus, some of the foundations of the rules of design, which were for the most part only theoretical, in a more scientific manner than had hitherto been done. These experiments were conducted in presence of six Dutch engineers, an experimenter in charge of the Harkort Works, and a number of representatives of Friedrich Krupp and of the Union of Dortmund.

Also a number of chemical analyses were made. They are given in Table No. 1. But inasmuch as the Dutch engineers did not carry out their intention of furnishing 90

more themselves, they are not complete. However, Table No. 1 will give an idea as to the chemical qualities of the material used.

TABLE No. 1.

Chemical Composition in Parts of One Per Cent.

	Hard Steel.	Medium Steel.			Ingot Iron.
Steel taken from Girder No....	16	30	27	28	6
Tensile strength..........	120,600	94,300	84,400	91,500	68,700
Carbon..................	0.370	0.294	0.260	0.365	0.120
Phosphorus..........	0.112	0.405	0.154	0.105	0.103
Copper..................	0.059	0.119	0.126	0.160	0.095
Sulphur..................	0.055	0.058	0.082	0.093	0.030
Manganese...............	0.285	0 072	0 253	0.237	0.271
Silicon..................	0.536	0.070	0.014	0.018	0.027

Of the 32 girders, 23 were of steel, which was specified to stand 85,332 lbs. per square inch (60 kilos per square millimetre), with an extension of 17 per cent after rupture and a contraction of area of 25 per cent. Plates of 0.28, 0.36, and 0.4 inch thickness should endure without cracks bending under pressure in previously prepared blocks to the corresponding angle of 140, 120, or 110 degrees. The extension was measured on specimens which were 8 inches long between points. Fifteen out of these 23 girders were selected at random from the girders for the Nymwegen bridge.

Three girders were made of hard steel, specified to the ultimate strength of 113,776 lbs. (80 kilos per square millimetre), with the previous conditions for bending.

Three girders were built of homogeneous (ingot) iron furnished by Fr. Krupp. It was ordered to stand from 71,110 to 64,000 lbs. per square inch, to stretch 25 per cent, and to bend as already specified. This class of material is probably not unknown to many Americans, since Thomas Prosser, of

New York, distributed pamphlets of the records of experiments made by David Kirkaldy in comparison with the best brands of Yorkshire iron, which, according to those experiments, it considerably exceeds in quality.

Finally three girders were made of iron, puddled and rolled at the Harkort Works, specified to stand 57,500 lbs. per square inch (36 kilos per square millimetre), to stretch 8 per cent, to bend to 50 degrees, and to show 13 per cent contraction of area.

It will be seen that as a rule the material of the girders tested exceeded the quality specified.

In a strongly built direct-acting lever testing-machine, capable of exerting tensions of 22 tons, three tensile tests of the metal of each of the above girders were made. The test girders were cut from the *broken* girders, one from the end, the second from the centre of the web-plate, and the third from the centre of the covering flange-plate.

The construction of the girder testing-machine exhibits the usual application of force by hydraulic pressure and measurement by a combination of levers. The weight of the scale consists of water in a cylinder, the level of which is indicated by a glass tube. One millimetre of rise of level corresponds to the addition of one kilogram of water. The hydraulic press is handled in such a manner that the cylinder is but very little above a stone support provided with the knob of an electric bell which rings in case the cylinder should touch the bearing. In this manner the knife-edges are kept in their correct position. The machine is constructed to exert pressures up to 250 tons.

The pressure of the press acts upon the centre of the tested girder. Hence the shearing force of the girder is the same in any of its sections. Therefore the rivets are more exerted in the centre of the girder than would be the case if double the pressure of the press were uniformly distributed over

the girder, though in both cases the maximum moments would be equal.

In order to transfer those concentrated pressures, four stiffening angles are bolted to the stringers at the ends as well as in the centre.

The results of the experiments were represented on 32 lithographed tables accompanied by 10 diagrams showing the curves, which are obtained by laying down the deflections as ordinates corresponding with the maximum strains per square unit by which they were produced.

In the calculation of strains the weights of the various parts of the testing-machine, of the apparatus for measurement of deflections, and that of the girder itself were duly considered.

The moments of resistance of the different girders were also carefully calculated, the central sections, with deduction of the rivet-holes, serving as basis of comparison.

If it is required to calculate the moduli, it is essential to use a moment of inertia between the two, one calculated with and the other without the rivet-holes. The writer has calculated from the figures of the tables the moduli in pounds per square inch. But as in these tables the moments of inertia *without* the rivet-holes are used, the moduli became much too great.

The deflections were read very frequently. In some instances 100 and more readings were taken, so as to observe the deflections for each additional kilogram of strain per square millimetre (1422 lbs. per square inch).

In cases when a sudden increase of deflection took place, or when a loud sound or report was noticed, the water was made to run out, the girder was carefully examined, and the experiment commenced again, when naturally a small permanent set was observed. Thus in one instance the experiment was repeated six times. The M_1, M_2, . . . M_6 of Table

2 refer therefore to the moduli as obtained during the first, second, . . . sixth repetition of the experiment.

The tables give the deflections in hundredths of one millimetre ($\frac{1}{2500}$ inch).

The tested girders were of the following construction:

Type 4–5.—Steel stringers of the 426-foot span of the Nymwegen bridge. Web, 27 in. by 0.28 in.; four angles, $2\frac{3}{4}$ by $2\frac{3}{4}$ by 0.28 in. (the legs with parallel sides, equal thickness); length between knife-edges, 16 ft. 5 in. One tensile or bottom flange-plate, $5\frac{15}{16}$ in. by $\frac{5}{16}$ in.; moment of inertia in centre, 1371 (1734 with area of holes considered); moment of resistance, 91.4 cubic inches (or 115.6).

Type 5–6.—Steel stringers of the 426-foot span. Web, $26\frac{3}{4}$ in. by 0.28 in.; four angles as before; length between knife-edges, 17 ft. $8\frac{1}{2}$ in. by 4 in.; top and bottom flange-plates, $5\frac{15}{16}$ in. by 0.28 in., the latter only resting over a part of the girder; moment of inertia, 1624 (1995); moment of resistance, 118 (145).

Type 6–7.—Steel stringers, same as 5–6; length, however, 19 ft. The rivets of these three types were $\frac{11}{16}$ in. in diameter, spaced $3\frac{1}{8}$ in. apart.

Type Floor-beams.—Web, $39\frac{3}{8}$ in. by 0.36 in.; four angles, $3\frac{9}{16}$ by $3\frac{9}{16}$ by 0.4 in.; length, 26 ft. 3 in. between knife-edges; rivets, $\frac{13}{16}$ in. in diameter, spaced $3\frac{9}{16}$ in.; two top and two bottom plates, $8\frac{11}{16}$ in. by 0.4 in., of which two extend all over the beam and two are only 16 ft. 5 in. long; moment of inertia, 9080 (10,943); moment of resistance, 443 (534) cubic inches.

Type of Iron Girders.—Stringers like 6–7, but built much stronger. Each consisted of a web-plate 26.9 in. by 0.4 in; four angles, $2\frac{3}{4}$ by $2\frac{3}{4}$ by 0.4 in., and three top and three bottom flange-plates, 6.7 in. by 0.24 in. Two of these plates reached over the whole girder; the others were shorter according to theory.

The moment of inertia was 3551 (4390), and the moment of resistance 246 (304) cubic inches.

The webs of the girders were stiffened by angles filling only the free space between the flange-angles. The temporary stiffeners, however, reached up and were fitted between the horizontal legs of the flange-angles. They rested on flat bearing-plates.

The web-plates of the floor-beams were spliced in the centre with pairs of plates 0.28 in. thick each.

Of the 32 girders tested, the following different classes must be considered separately:

1. Experiments (Nos. 5, 16, 24) with stringers of type 6–7 were built of hard steel furnished by the Rhenish Steel Works, and of steel rivets of the usual Dutch Government quality taken from stock. "The holes were drilled at once through angles and plates; they were not reamed nor drift-pinned; during riveting the parts were held together by temporary bolts, and in every regard the girders were built up with the greatest possible care."

2. Experiments with stringers of medium steel (85,332 lbs. per square inch), namely:

Nos. 12, 13, 14, 15 of type 4–5, } all six from Nymwegen
Nos. 1–2 " " 5–6, } bridge;

Nos. 3, 7, 8, 9, 10, 11, 19, 22, 27 of type 6–7, six of which were from Nymwegen bridge.

The steel for these 15 stringers was furnished by the Union works.

"All holes were bored, reamed smooth, the material pickled and oiled before riveting, hence the whole manufacture of the usual bridge-builders' work."

3. Experiments also with medium steel of the Union. The girders Nos. 30, 31, and 32, however, were floor-beams. Manufacture the usual one as under 2.

4. Experiments 20, 21, 26, type 6–7, of medium steel from

the Dortmund Union, but previously annealed at the steel-works. Manufacture the same as under 1.

5. The stringers 28 and 29, also of medium steel from the Union, were not riveted but bolted together.

"All holes were drilled through plates and angles at once, then reamed slightly conical; thereupon the bolts, likewise conical, were placed and firmly, but without force, screwed home. Drift-pinning was entirely avoided, only temporary bolts being used; in all respects the girders were well built up for the purpose of the experiment."

6. Experiments 6, 17, and 25 were made with stringers type 6–7, built of homogeneous (ingot) iron. The steel rivets were of the usual Dutch Government stock. Manufacture the same as under 1 and 4.

7. Finally, the experiments 4, 18, and 23 were made with the three puddled iron stringers with flange-plates. The rivets were of the Dutch quality from stock, and the manufacture extra good, as under 1, 4, and 6.

The elastic qualities of the tested girders can be studied from Table No. 2.

To show clearly these qualities, there is perhaps no better method than to calculate the *average* moduli, as obtained from the deflections caused, first, by a starting-strain, and then by successive greater test-strains. A starting-strain is adopted in order to eliminate occasional disturbances caused by slight local depressions, etc.

For the purpose of this abstract, the rich material of the tables was condensed. One column of the "Original Moduli Mo" gives the average values between strains only caused by the dead load, etc., of the girder and a maximum strain of 14,222 lbs per square inch (10 kilos per square millimetre). These values (Mo), therefore, refer to the behavior of girders in practice. They show with what degree of justification in certain kinds of scientific investigations the modulus may be supposed to be a constant quantity.

Table No. 2.

Average Moduli in Millions of Pounds per Square Inch.

		Hard Steel.			Medium Steel, usual good Manufacture.					
		Stringers.			Stringers.					
		Type 6–7.			4–5				5–6	
Number of Experiment		5*	16*	24*	12	13	14	15	1	2
Original moduli up to S = 14,222 lbs. Mo =		34.7	33.0	34.7	41.0	38.6	39.8	43.0	36.5	38.1
The average moduli are taken between the following limits of maximum strains:	Pounds per square inch.	M_1	M_3	M_1	M_1	M_1	M_1	M_1	M_1	M_1
	7,111 and 14,222	31.7	31.7	31.7	38.8	32.4	31.1	40.8	30.5	35.4
	7,111 and 21,333	30.6	31.3	30.6	39.0	34.7	33.1	41.0	28.5	32.7
	7,111 and 28,444	30.6	31.6	30.6	35.0	33.6	33 1	39.1	32.3	32.1
	7,111 and 35,555	30.3	31.4	30.3	34.8	32.8	33.6	36.0	33.1	30.7
	7,111 and 42,666	30.6	31.6	30.4	33.8	32.8	34.0	37.0	32.8	30.4
	7,111 and 49,777	30.9	31.4	30.9	33.6	33.4	30.8	36.5	28.3	30.3
	7,111 and 56,888	30.3	31.1	30.3	32.8	33.5	27.0	36.4	18.6	29.7
	7,111 and 63,999	30.3	29.1	30.3	23.3	......	24.2	35.8	15.7	27.4
	7,111 and 71,110	30.3	30.7	30.3	21.4	......	......	35.4	......	24.7 (5–48)
	7,111 and 78,221	25.8	30.0	29.0	......	......	......	35.0	......	......
	7,111 and 85,332	20.6	29.0	25.0	......	......	......	33.8	......	......
	7,111 and 92,443	......	25.7	23.6	......	......	......	......	......	......
	7,111 and 99,554	......	22.4	20.6	......	......	......	......	......	......
Average moduli Mo		34.1			40.6				37.3	
And corrected Mo		28.6			33.3				31.3	
Elastic limits		78,200	89,500	89,500	56,900	51,400	48,500	64,200	48,600	48,600
Average limits		85,700			....				47,900†	
Av. ultimate strength of material		120,000			....				94,200†	

Note.—The figures (5–48), (5–42), etc., indicate that the girders broke before the next higher strains of 50 (71,110), 45 (63,999) kilograms per square millimetre were reached. The moduli are taken between the figures 5–48, 5–42, etc., kilograms.

* Specially built girders; the others are taken from those intended for the Nymwegen bridge.

† With the test on page 181.

TABLE No. 2.—*Continued.*

	Medium Steel, usual good Manufacture.									Medium Steel.		
	Stringers.									Floor-beams.		
	Type 6–7.									Flange-plates.		
Number of Experiment.....	3	7	8	9	10	11	19*	22*	27*	30	31	32
Original moduli up to S = 14,222 lbs. Mo =	38.5	33.5	32.8	36.8	36.1	39.2	33.0	40.2	35.4	21.7	20.4	26.7
The average moduli are taken between the following limits of maximum strains: Pounds per sq. in.	M_1	M_1	M_1	M_1	M_1	M_1	M_3	M_4	M_3	M_4	M_3	M_3
7,111 and 14,222..	38.0	32.7	30.8	33.5	40.0	35.8	31.7	33.5	34.5	26.0	24.5	28.1
7,111 and 21,333..	31.0	31.2	30.5	32.7	31.0	33.5	33.1	32.5	34.1	25.7	25.0	26.8
7,111 and 28,444..	30.1	**31.1**	30.8	33.2	31 1	**34.2**	39.1	32.5	34.8	25.7	24.8	26.4
7,111 and 35,555..	29.7	28.0	30.2	33.1	30.8	28.1 (5–24)	28.8	32.5	34.2	**25.0**	25.0	26.4
7,111 and 42,666..	**29.8**	25.7	**29.8**	33.7	**29.1**	......	?	33.1	34.0	24.5	**24.4**	**26.1**
7,111 and 49,777..	28.3	24.0	25.7 (5–32)	**32.5**	26.0 (5–33)	......	**32.5**	**32.4**	**33.0**	23.0 (5–33)	23.5	24.2
7,111 and 56,888..	25.2	20.0	......	28.2 (5-38)	......	......	30.7	30.0	32.1	......	23.2	24.5
7,111 and 63,999..	......	......	......	......	......	......	29.8 (5–42)	27.1	31.2 (5–43)	......	......	......
7,111 and 71,110..	......	......	......	......	......	......	......	......	......	......	......	......
7,111 and 78,221..	......	......	......	......	......	......	......	......	......	......	......	......
7,111 and 85,332..	......	......	......	......	......	......	......	......	......	......	......	......
7,111 and 92,443..	......	......	......	......	......	......	......	......	......	......	......	......
7,111 and 99,554..	......	......	......	......	......	......	......	......	......	......	......	......
Av. moduli Mo.	36.2									22.9		
And corrected Mo...........	30.4									19.7		
Elastic limits...	44,000	31,400	42,700	52,600	42,700	34,100	50,000	50,000	52,600	38,200	42,700	42,700
Av. limits......	47,900									40,900		
Aver. ultimate strength of material......	94,200									92,400		

* Specially built girders.

TABLE No. 2.—*Continued.*

	Medium Steel.					Homogeneous.			Puddled Iron.		
	Annealed.			Taper bolts.		Ingot iron.			Stringers with flange-plates.		
	Type 6–7			6–7		6–7					
No. of Experiment....	20*	21*	26*	28*	29*	6*	17*	25*	4*	18*	23*
Original moduli up to S = 14,222 lbs. M =	22.4	33.0	35.7	40.0	36.5	34.1	29.7	31.2	33.4	34.1	34.7
The average moduli are taken between the following limits of maximum strains: Pounds per sq. inch.	M_1	M_2	M_1	M_2	M_2	M_1	M_4	M_2	M_1	M_2	M_6
7,111 and 14,222..	20.1	31.7	30.0	33.8	32.1	28.5	31.7	29.2	29.5	35.2	33.1
7,111 and 21,333..	**19.5**	30.8	28.1	32.1	32.2	27.4	30.7	23.9	30.4	34.2	32.5
7,111 and 28,444..	17.1	**31.1**	26.2	27.4	32.8	**26.5**	**31.7**	**30.2**	**31.0**	**34.7**	**32.7**
7,111 and 35,555..	11.4 to 10.2	20.5 (5–23)	**25.2**	27.4	32.0	25.2	31.0	30.2	28.5	31.2	32 8
7,111 and 42,666..	......	......	24.1 (5–28)	30.8	**30.7**	20.0	30.4	24.2	21.7	24.8	33.2
7,111 and 49,777..	......	......	......	**29.5**	28.5	11.2	22.0	13.8	11.7	21.4 (5–32)	15.5
7,111 and 56,888.	......	......	......	26.8	25.2	......	12.0	......	8.0 (5–37)	.....	7.0 (5–39)
7,111 and 63,999	......	......	......	20.8	19.5	......	......	......	......	......	......
7,111 and 71,110..	......	......	......	16.8	.....	......	......	......	......	......	.. ...
7,111 and 78,221..	......	......	......	......	......	......	......	......	......	......	......
7,111 and 85,332..	......	......	......	......	......	......	......	......	......	......	......
7,111 and 92,443..	......	......	......	......	......	......	......	......	......	......	......
7,111 and 99,554..	......	......	......	......	......	......	......	......	......	......	......
Average moduli Mo..	30.3			38.2		31.7			34.1		
And corrected Mo....	25.4			32.0		26.6			28.6		
Elastic limits	19,000	31,300	35,600	50,000	42,700	33,000	33,000	33,000	28,400	28,400	28,400
Average limits........	28,600			46,300		33,000			28,400		
Av. ultimate strength of material..........	72,000			91,000		65,400			55,500		

The other moduli are all calculated from the same starting-strain of 7111 lbs. (5 kilos per millimetre), and also from twice, three times, four times, etc. This same strain (between 5 and 10, 5 and 15, 5 and 20, . . . to 5 and 65 kilos per millimetre.

In 14 vertical columns of the table the moduli were calculated from the *last* repetition of experiment with each girder. Hence the moduli M_2, M_3, . . . M_6 are somewhat *greater* than would have been the case if only the first experiment had been made and continued to the destruction of the girder.

The next column contains the average original moduli of each class; as already explained, these moduli are all too great. Since, however, not their absolute but their *relative* values are most interesting, a new calculation with the moments of inertia with consideration of the area of the rivet-holes was not made. But for the average values (Mo) of each class this was done. For this purpose 0.8 of the differences of the moments of inertia are added to the smaller ones, and therefrom the new moduli calculated.

The next column contains the *elastic limits.* Wherever the moduli of the previous columns commence to decrease sensibly for a specimen, the limit of elasticity must be looked for. In determining the same for the moduli of Table 2, the deflections and the curves of the diagrams were consulted together. Those moduli of the table which are characteristic for the neighborhood of the limit are indicated by bolder figures.

The table admits of some interesting conclusions.

The modulus and hence the deflections of a girder depend not only on the material, but also on design and manufacture. Thus the floor-beams, though of the same steel as the 15 preceding stringers, exhibit considerably lower moduli; also the elastic limits are lower than those of the 15 stringers. This must be due to the multiplicity of parts and rivets.

On the other hand, the girders built of puddled iron, though consisting of more parts than the floor-beams, probably by reason of their more careful manufacture, and also because of the greater uniformity of the material of their constituent parts, not only exhibit higher moduli, but also higher relative limits. Their diagrams also show much greater regularity and uniformity of the curves of deflection.

Annealing did not prove of any benefit with the three steel stringers Nos. 20, 21, and 26. It seems to have reduced the moduli as well as the elastic limit, which latter on the average is only 28,600 lbs., as against 47,900 of the 15 stringers of the same kind of steel. The curves of deflection of the annealed steel girders, notwithstanding their extra-good manufacture, differ considerably one from the other. It seems that the steel was not evenly annealed.

Striking by their regularity of curves of deflection, of their moduli and elastic limits are the two bolted girders, the three stringers of homogeneous metal, and those made of puddled iron.

The conical bolts in conically reamed holes give results which are equally as good as those by riveting; nay, compared with the 15 riveted stringers of the same kind of steel the bolted girders were much superior.

Also, the three girders of hard steel, probably because of their extra-good workmanship, with exclusion of drift-pins, gave good curves of deflection, regularly decreasing moduli, and regular and high elastic limits, the latter much enhanced by the hardness of the metal. The elastic limits of the *finished* girders are of special interest. They are the turning-points in the curves of deflection, near which the curvatures are sharper than anywhere else. For the bolted girders, the homogeneous metal stringers, and the puddled iron girders with flange-plates the limits are almost exactly

one half of the ultimate resistance of their material, corresponding with the usual limit of elasticity of ***unmanufactured*** iron. The behavior as regards deflections beyond that limit is known to be entirely dependent on the specific nature of the material used, and the same observation must be made for the finished girders. Now the diagrams of deflections, as well as the moduli of the bolted steel stringers, of those of homogeneous metal, and of the iron girders, agree strikingly with the *forms* of curves as known to exist for the best classes of iron and steel, tested in their original condition, such as for instance are shown in Styffe's work. Hence the 32 experiments before us do not present the remotest endorsement for a certain rule, recently alleged in an English engineering periodical, that girders under pressure exhibit striking deflections at the $\frac{3}{4}$ point of their ultimate strength, or that they fail suddenly or by compression. On the contrary, the behavior of these 32 experiments, which by their thoroughness and scientific nature put far into the shade the few old ones, prove once more that beyond that limit the possibility of any scientific rule ceases. The elastic limit itself is the turning-point beyond which the greatest variety may be found, according to more or less complex construction, according to good or bad workmanship, and according to the quality of the material, of which one extreme is perhaps represented by the behavior of the poorest description of rolled girders, such as not infrequently are used in English architectural iron-work, and the other extreme is presented by experiments with the finest Swedish charcoal-iron.

For the study of the ultimate strength of the materials of the 32 girders, and of their failure, Table No. 3 was condensed from the reports of the experiments.

Table No. 3.

Strength in Pounds per Square Inch of Net Area.

Type of Girders.	Hard Steel (80 ko).			Medium Steel (60 ko).					
	6-7			4-5				5-6	
Manufacture.	Extra, *no drift-pins*, riveted.			Ordinary good manufacture, drilled and reamed holes.					
Number of Experiment......	5*	16*	24*	12	13	14	15	1	2
Web-plates. Ultimate tension........	118,700	120,000	116,600	88,900	88,200	96,700	94,600	99,600	95,900
Extension, per cent.....	14	14	14	16	14.5	16	13	15.5	16
Extension, per cent of non-contracted part....	19.5	10	10	11	11	11	9	11	12
Contraction, per cent....	30.5	32	35	33.5	27	35	35	.34	39
Bent to angles of degrees	102	73	106	180	162	132	93	148	116
Flange-plates. Ultimate tension...	119,500	120,900	119,400	91,000	83,900	......	89.600	89.600	88,100
Extension, per cent.....	14	14	14	19	18	......	15	18	17
Extension, per cent of non-contracted part....	5	10	10	14	14	......	11	14	15
Contraction, per cent....	31	24	36	35	33	......	31	31	33
Bent to angles of degrees	180	109	136	.....	......	......	......	150	150
Strains of first rupture of tension-flanges................	78,200	....	....	... ..	52,600	49,800	83,900	55,400	42,700
Strains of total failures.......	96,700	96,700	83,900	69,700	54,000	66,800	83,900	65,400	68,200
Average breaking strains....	86,300			50,900					
Ultimate tension of metal failing first	120,000	120,900	120,000	91,000	93,800	98,000	89.600	89,600	88,200
Percentage realized..........	65.2	80	70	76.5	56.1	50.8	93.6	61.9	48.4
Aver. percentage of strength realized....................	71.7			56.1					

* Specially built girders.

TABLE NO. 3.—*Continued.*

Type of Girders.		Medium Steel (60 ko). 6–7									Medium Steel. Floor-beams, Flange-plates.		
Manufacture.		Ordinary good manufacture, drilled and reamed holes.									Usual, drilled and reamed holes, riveted.		
Number of Experiment....		3	7	8	9	10	11	19*	22*	27*	30	31	32
Web-plates.	Ultimate tension..	105,200	106,000	86,800	101,700	100,300	87,400	91,700	76,100	93,900	95,300	99,600	90,300
	Extension per cent.	14	14	16	15.5	12	16	15	17	17	14	16	18
	Extension per cent of non-contracted part..	10	11	12	12	8	13	10.5	11	12	10	11	13
	Contraction, p.c.	33	31	39	37	33	33	35	45	35	32	38	39
	Bent to angles of degrees..	145	137	133	148	146	180	180	180	162	67	99	120
Flange-plates.	Ultimate tension..	89,600	79,600	92,400	96,700	92,400	78,200	86,800	86,800	99,600	92,400	78,200	106,600
	Extension per cent.	16	18	15	19	20	20	13	16	14	17	16	14
	Extension per cent of non-contracted part..	12	12	11	10	13	12	11	12	11	13	11	10
	Contraction, p.c.	36	48	34	39	39	41	36	25	......	42	46	43
	Bent to angles of degrees..	170	180	74	85	90	149	94	180	178	86	165	68
Strains of first rupture of tension-flanges.....		25,600	32,700	......	....	....	34,100	54,600	54,000	......	44,500	......	....
Strains of total failures.....		56,900	58,300	45,500	54,000	47,000	46,800	61,000	64,000	62,000	48,300	58,600	56,900
Aver. breaking strains..		50,900									53,300		
Ultimate tension of metal failing first..		89,600	79,600	92,400	96,700	92,400	78,200	90,000	86,700	99,500	92,400	78,200	106,600
Percentage realized.......		28.6	41.0	49.2	55.8	50.8	43.6	60.6	62.3	62.0	48.1	74.4	53.4
Average percentage of strength realized......		56.1									58.6		

* Specimens especially built for the experiments.

TABLE NO. 3.—*Continued.*

Type of Girders.	Annealed Medium Steel.			Medium Steel.		Homogeneous Ingot Iron.			Puddled Iron.		
	6-7			6-7		6-7			Stringers, 6-7 Flange-plates.		
Manufacture.	Extra, *no drift-pins*, riveted.			Extra, *no drifts*, bolted.		Extra, *no drift-pins*, riveted.			Extra, *no drift-pins*, riveted.		
No. of Experiment...	20*	21*	26*	28*	29*	6*	17*	25*	4*	18*	23*
Web-plates. Ultimate tension.	88,200	71,800	74,100	89,600	86,700	69,400	59,700	63,300	56,200	56,200	56,900
Extension, p. c..	16	18	18	15	14	22	20	24	22	22	24
Extension, p.c. of non-contracted part............	13	14	13	11	10	13	14	18	......	......	......
Contraction, p. c.	33	35	35	34	35	43	46	46	27	31	30
Bent to angles of degrees	136	180	176	160	147	180	180	180	140	126	133
Flange-plates. Ultimate tension.	65,400	71,100	78,200	95,300	96,700	68,300	68,300	68,300	54,000	56,900	55,500
Extension p. c...	5	15	16	13	13	21	24	24	15	14	12
Extension p. c. of non-contracted part............	......	8	12	10	10	14	19	18	14	10	......
Contraction, p.c.	......	37	27	20	26	47	49	50	23	24	11
Bent to angles of degrees........	128	29	28	180	177	180	180	180	176	180	115
Strains of first rupture of tension-flanges	24,200	......	28,200	......	......	......	......	......	......	......	49,800
Strains of total failures.	36,900	32,700	41,000	71,000	68,000	51,000	50,000	50,000	54,000	55,500	55,900
Average breaking strains..............	28,400			69,500		50,300			51,900		
Ultimate tension of metal failing first...	65,400	71,000	78,200	91,500	90,000	68,700	62,600	64,800	54,000	56,900	55,400
Percentage realized..	37	46	36	77.6	75.5	74.2	80.5	77.1	100	97.5	90
Average percentage of strength realized	39.7			76.5		77.7			95.8		

Of the hard steel girders two failed by rupture of the tensile flanges in or near the centres of the girders. No. 16 failed by crippling of the compressional flange, a fold also being formed in the web-plate, both failures being found between the additional stiffeners in the centre and the nearest permanent stiffener. The girder did not fail suddenly, but gradually, and it was impossible to obtain any higher pressure by pumping.

Thr first rupture of No. 5 happened in both its tensile flanges, which had their horizontal legs torn.

The high percentage of 71.7 realized by hard steel stringers must be attributed to their extra-good manufacture, with exclusion of drift-pins.

As regards the 15 stringers of medium steel, their material was uniform enough, and the works, knowing it to be intended for the Dutch bridge, certainly tried to give satisfaction, and 12 out of these 15 girders were originally intended for the Nymwegen bridge. All 15 girders failed in tension, in most instances at their centres, or at most only one, two, or three rivet-distances from the centre, and the rents, as in all analogous cases, ran through the rivet-holes. All these 15 stringers were manufactured by the Union of Dortmund.

Three of these stringers were so placed in the testing-machine that the compressional flanges only consisted of two angles. Girder No. 14, on the contrary, was placed in a reversed position, its tensile flange consisting of two angles.

Ten of the girders were already broken by strains lower than those under which they were entirely ruptured. Of experiments 14, 2, 19, and 22, these ruptures happened through the flange-angles alone; in the other instances through the flange-plates, with or without broken angles. Probably after these first ruptures, owing to the hardness of the material, the neutral axis was moved near to the compression-flanges, thus enabling the stringers for a short time to carry addition-

al loads. For practical purposes, however, the girders, after their first ruptures in the tensional flanges, would have to be considered as unfit for further use.

For this reason the writer calculated the percentage of ultimate resistance from these first tensile breakages.

The ultimate strength of this lot is very irregular and unexpectedly low, being less than the strength of the iron stringers, though these are of much more complex design. Some of the 15 stringers behaved no better than would have been expected from good cast-iron. If the ultimate strength were calculated from the total rupture of each girder, 66.3 per cent of the calculated strength would be found as the average net result realized, instead of the 56.1 per cent of the table. The ultimate strength itself would be 60,200 instead of 50,900 lbs. The three floor-beams of medium steel and of good usual manufacture (built at the Union Works) also broke through their tensile flanges in the centres of the girders. Their results are somewhat more regular and better. The three beams and 15 stringers together show an average strength of 51,900 lbs., or 56½ per cent of their calculated strength, the rivet-holes being deducted.

The annealed steel girders also broke by tension, the result being still more unsatisfactory, not 40 per cent being realized. In report No. 20 it is mentioned that the rupture of the tensile test-piece from the flange exhibited burned crystalline structure.

The influence of careful manufacture is plainly set forth by the two experiments with stringers built together with tapering bolts and without drift-pins. Both these girders, though of the same kind of steel as the 21 preceding ones, failed not by tension as the others, but by crippling of their compressional flanges at places near to the first permanent web-stiffeners in the centre. They did not fail suddenly, but gradually like No. 16 of hard steel. The three homoge-

neous metal girders failed through folds of their webs and bulging or crippling of the compressional flanges. The girders 6 and 25 exhibited two regular web-folds, one at each side of the temporary central stiffener. The three girders again failed very gradually, like Nos. 16, 28, and 29. The failure of No. 25 was accompanied by continued crackling of the bending parts, its tensile flange also becoming much rounded.

This isotropous metal, undoubtedly of excellent qualities for boilers, seems too soft and yielding for bridge-girders. Not quite 78 per cent of the theoretical strength was realized, which is less, absolutely as well as relatively, than yielded by the much cheaper puddled iron.

Of the last group of three puddled-iron girders with three pairs of flange-plates, No. 4 can hardly be considered as having totally failed. It was of course permanently bent, and the compressional flange had assumed a wavy appearance between the rivets as fixed points. Nor were the two other girders broken down anything like the girders of medium steel and usual manufacture. No. 18 failed at the same time, not only by two regular web-folds symmetrically located near the central stiffeners, combined with a slight bulging of the flange, but also by numerous small tensional ruptures, in the other flange-plate these ruptures starting in a regular manner from the most strained parts of the rivet-holes. No. 23 principally failed by tension, and at the same time the web formed folds, and the compressional flange began to get wavy.

These three iron girders, whose regular and almost identical curves of deflection we have already mentioned, realized 95.8 per cent of their calculated strength, and surpassed all others in the regularity of their behavior. Such excellent iron, however, which in plates of one metre width and for 6¾-inch flats shows an absolute average strength of 56,000

lbs. (exactly 25 tons) on long specimens is not obtained easily everywhere. It was this good quality, still further evidenced by the uniform good fibrous fracture of the six tensile test specimens, and combined with the superior manufacture, which caused the excellent result. It may not be superfluous to add, that had tensile tests also been made on pieces of the angles, the original ultimate strength of the table would have become a little higher, inasmuch as angles, owing to the great pressure which they receive in the rolls, as a rule show higher ultimate strength than plates. Comparing the results of the 32 Harkort experiments with older ones, it may be remarked that it has been stated several times by so high an authority on steel as the late Dr. W. Siemens that the full theoretical strength of joints of riveted steel plates must not be expected, and that engineers have yet to learn how to use steel properly.

As regards the few old experiments on riveted iron girders, the writer, not satisfied with the interpretation by others, a number of years ago thoroughly studied the originals of those experiments. He found that they were incomplete; that the manner in which the experiments were conducted was not given in full; that the nature of the manufacture of the specimens, their tensile and other qualities were not recorded, etc. As a rule, a number of riveted tubes and the model of the Britannia Bridge (repeatedly broken, reinforced, patched, and re-tested) gave ultimate strength of about 35,000 lbs. per square inch of net area, or but little more. The girder tested by Brunel was the most unsatisfactory of all; it broke—though of exceptionally good iron—at about 12 tons per square inch of net area, and exhibited other irregularities which do not entitle its experimental results to the rank of scientific or trustworthy material.

Modern bridges are so designed that in their main parts the material is acted upon either by tension or by pressure.

Therefore what seems to be needed is not a delicate, sensitive, and often capricious, isotropic material of semi-crystalline structure, but a kind of material which is *homogeneous* and of *excellent* qualities in *one* direction only, of sufficient *stiffness*, and of sufficient lateral cohesion of its *fibres* to withstand the inferior secondary strains to which it may be subjected.

There is but very scanty experience in existence as regards steel bridges. But the experience with steel railway-axles is much more complete, and the writer in conclusion ventures to remark that the German Railroad Union, comprising nearly 36,000 miles of road, stipulates in its regulations that cast-steel axles may be strained 20 per cent higher than iron axles.

The results of the Harkort experiments, with numerous specimens of the broken girders, of the tensional test-pieces, diagrams, etc., were exhibited at the Dusseldorf exhibition of 1880, where the writer studied them for two days. He may state that he found nothing but what agrees with the reports of the experiments from which the above abstract was made. With exception of a short review in a report on the exhibition in *Glaser's Annalen*, so far as the writer knows, they have not been published before.

II.

MR. JAMES CHRISTIE'S EXPERIMENTS ON IRON AND STEEL STRUTS MADE AT THE PENCOYD IRON WORKS IN PENNSYLVANIA.

James Christie, Esq., of Pencoyd, has indebted the engineering profession by his very numerous experiments on the compressive strength of iron and steel struts, and on the transverse strength of rolled iron and steel beams.

The results were obtained with a Fairbanks lever testing-machine of 50,000 pounds capacity.

They were laid down and explained in two papers presented by Mr. Christie to the American Society of Civil Engineers in 1883 and early in 1884. They deserve to be carefully studied.

Of the extensive material contained in these two papers, some of the principal conclusions are as follows:

In agreement with older experiments, the great variability of the moduli of elasticity of iron and steel is once more confirmed. The tensile moduli, as obtained with the U. S. Government testing-apparatus at Watertown, varied from 24 to over 33 millions of pounds per square inch for iron. The compressive moduli of iron varied from 26½ to 35.3 millions of pounds. Tests on flexure yielded moduli from 20 to over 30 millions. The compression-moduli of steel were found 15 per cent lower than those for iron; so that if these results are confirmed by others, the compressional strength of struts of steel must decrease more rapidly than that of iron struts. The experiments have further shown that struts of hard steel are considerably stronger than those of mild steel (ingot-iron) or of puddled iron, so that it will be advisable for bridges of very great span to specify struts of a somewhat hard description of steel. Special tests will still be desirable in each separate instance.

Mr. Christie's experiments illustrate in a striking manner the great reduction of the strength of struts caused by eccentric application of pressures, and hence the very great importance of *careful* and *accurate workmanship*, both in the shops and on the scaffold. It would be desirable to make comparative experiments with iron and steel struts slightly curved for the purpose of the experiments, and also on the influence of eccentric application of pressures, so as to learn how the two materials compare in these respects.

Struts bearing on thick well-fitting pins were found as strong as flat-ended struts, but slight deviations from the central application of pressures reduced the strength considerably.

Long struts with *fixed* ends were found to be noticeably stronger than flat-ended or pin-bearing struts.

As regards the transverse strength of iron and steel beams, it was found that those of mild steel (carbon one eighth of one per cent) were 15 per cent stronger than iron beams. One single experiment made with a beam of hard steel (carbon one third of one per cent) gave 62,000 pounds, whilst the average for 13 iron beams was 43,000, and the average for 13 beams of ingot-iron was 49,400 pounds per square inch.

The original tensile strength of the iron was 51,000 pounds, of the mild steel 62,000, and of the hard steel 100,000 pounds.

The formula used in calculating the imaginary ultimate strains by flexure was the usual formula for flexures within the elastic limit. According to this mode of calculation, 84.3 per cent of the ultimate tensile strength of the iron, 79.7 per cent of that of the mild steel, and only 62 per cent of that of the hard steel were realized.

Plate I.a.

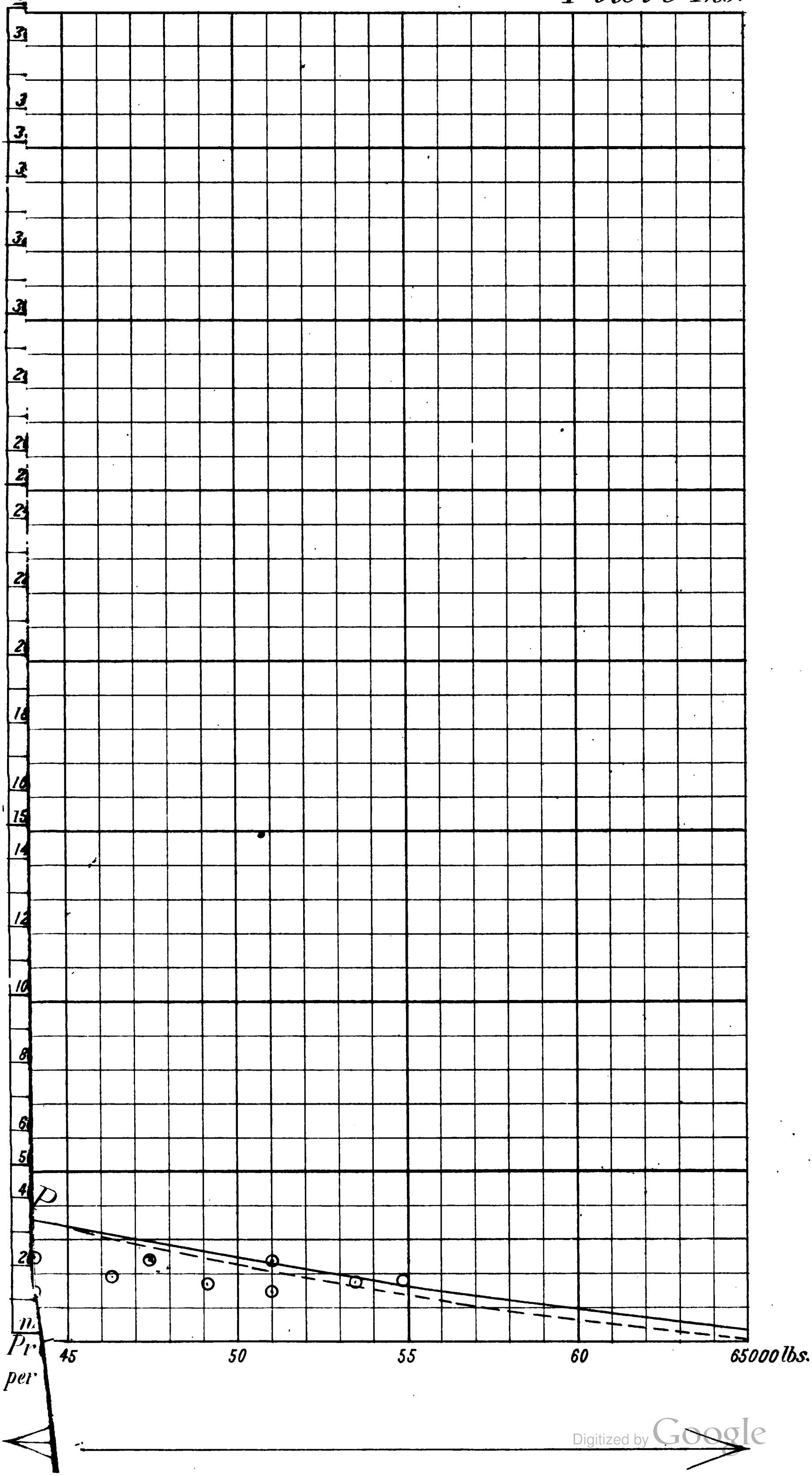


Plate I.b.

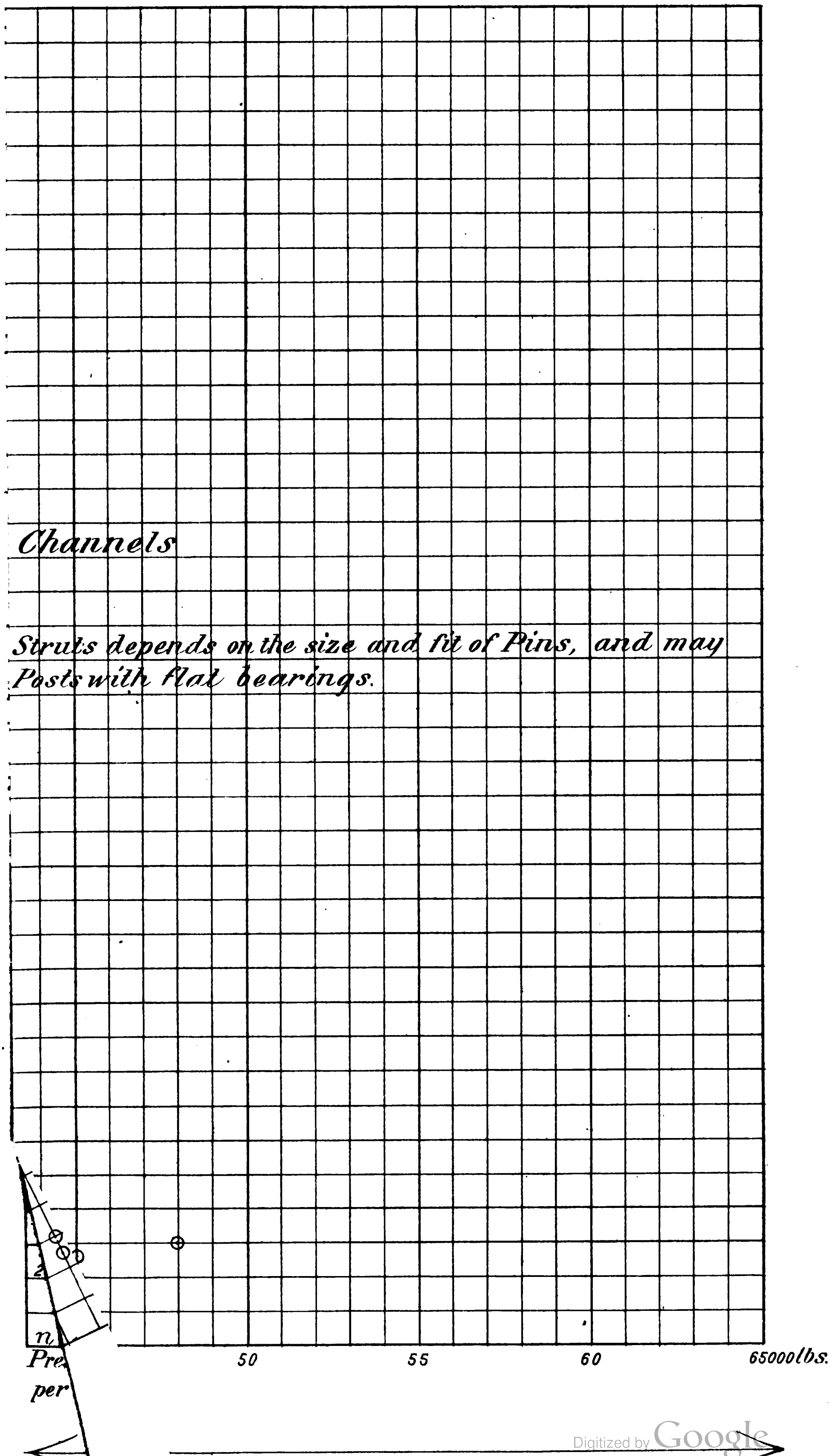

Channels
Struts depends on the size and fit of Pins, and may
Posts with flat bearings.
n
Pre
per
50
55
60
65000 lbs.

Plate I.c.

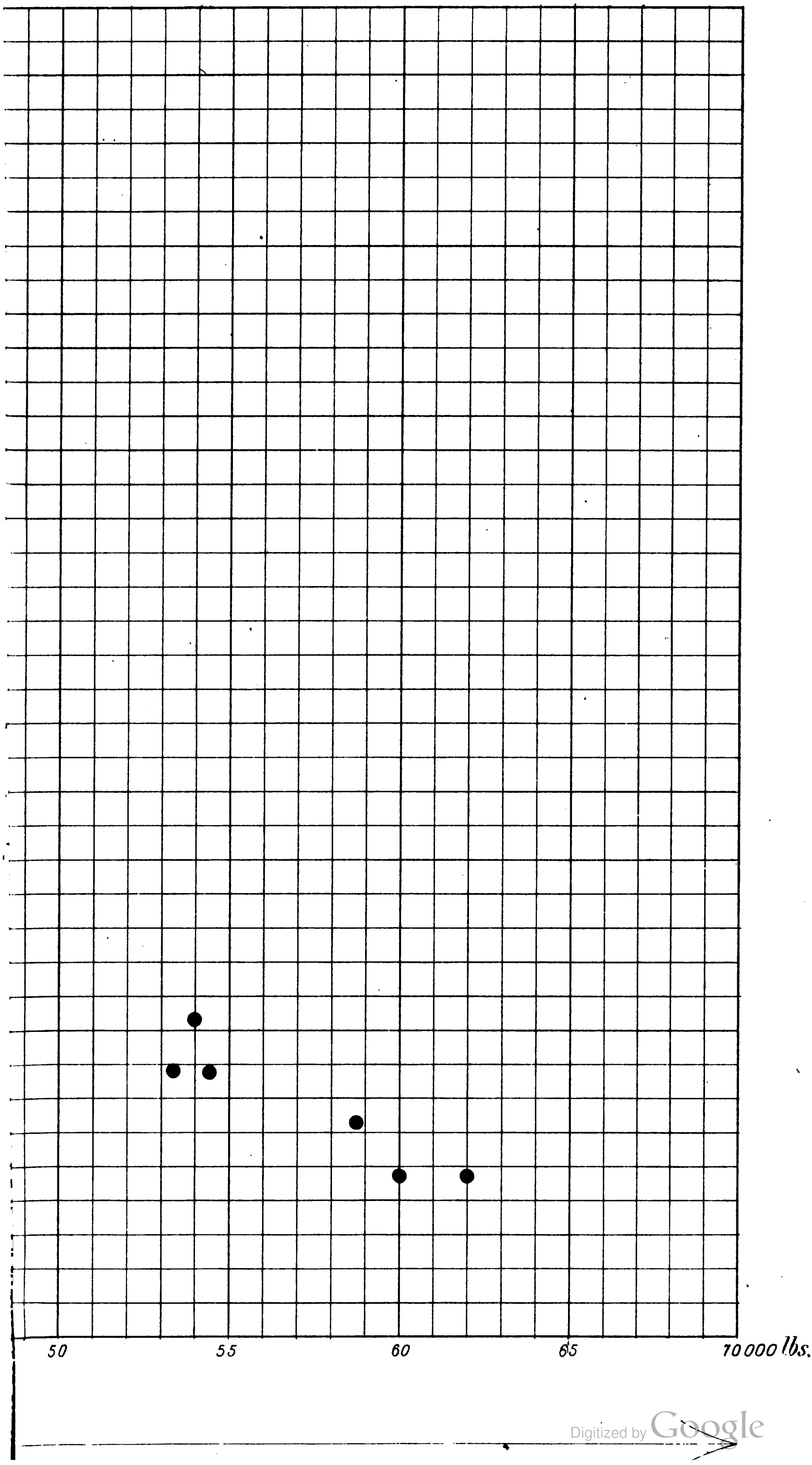


Plate I.

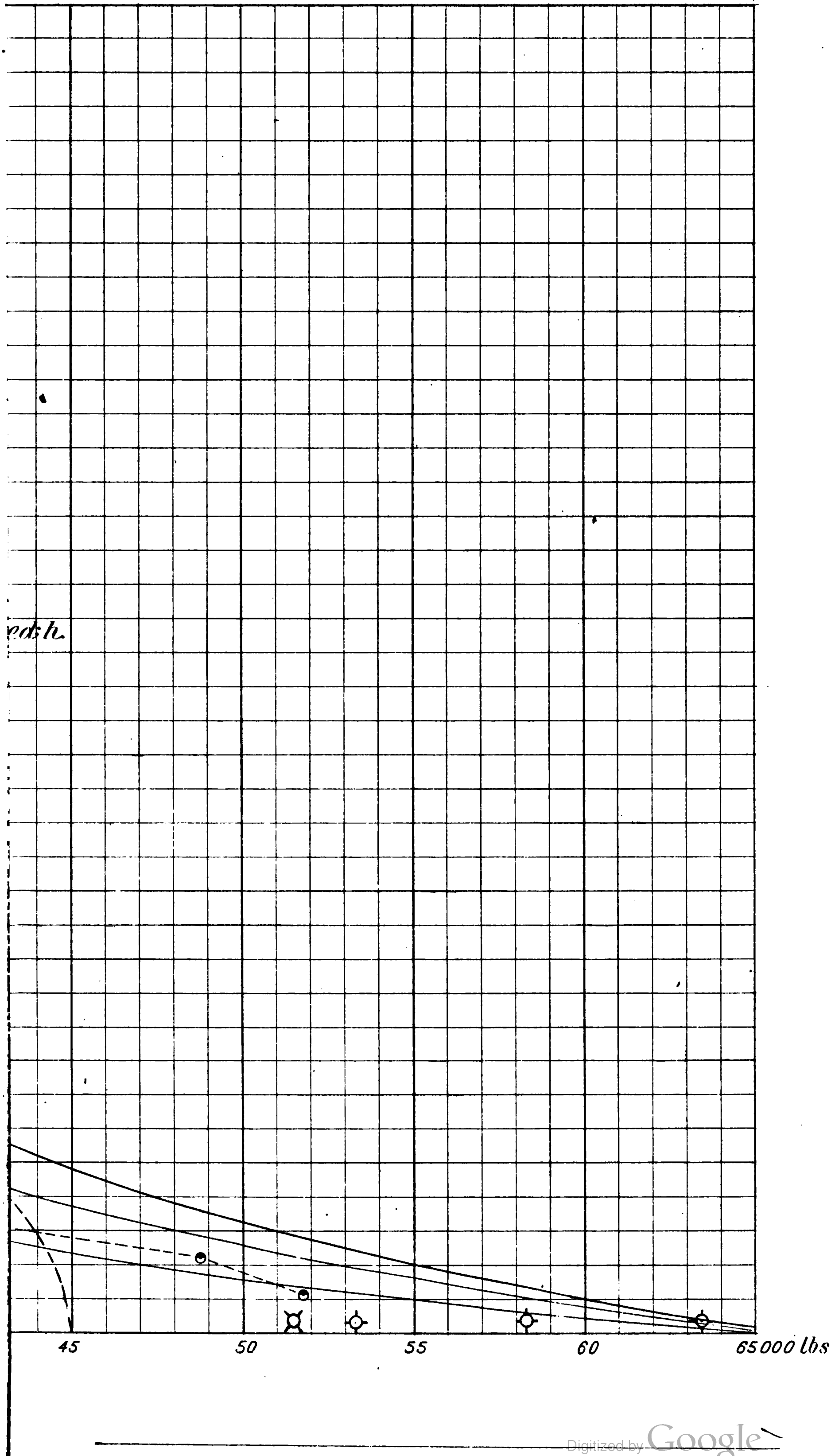


Plate II

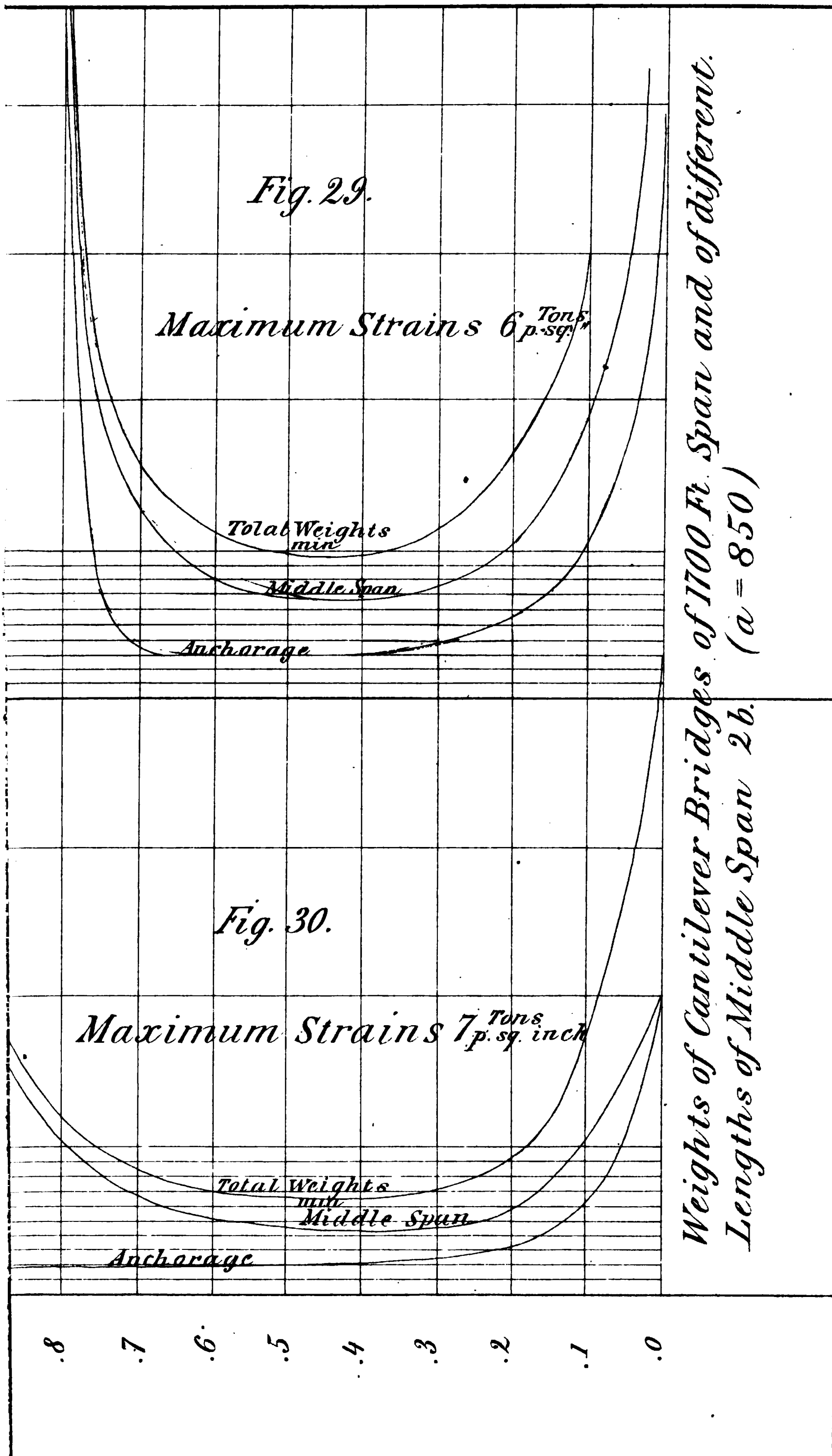


Fig. 29.
Maximum Strains 6 Tons p. sq."
Total Weights min
Middle Span
Anchorage
Fig. 30.
Maximum Strains 7 Tons p. sq. inch
Total Weights min
Middle Span
Anchorage
.8 .7 .6 .5 .4 .3 .2 .1 .0
Weights of Cantilever Bridges of 1700 Ft. Span and of different
Lengths of Middle Span 2b. (a = 850)

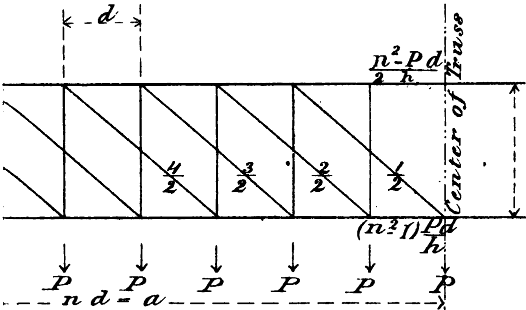


Fig. 3.

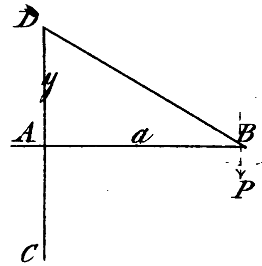


Fig. 7.

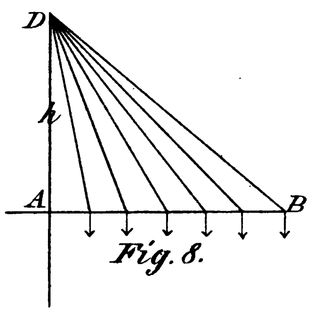


Fig. 8.

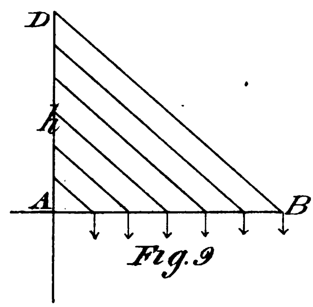
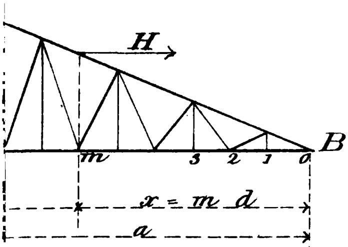


Fig. 9



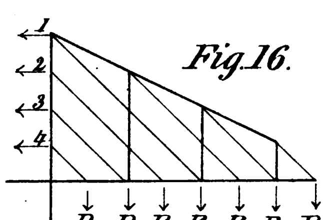
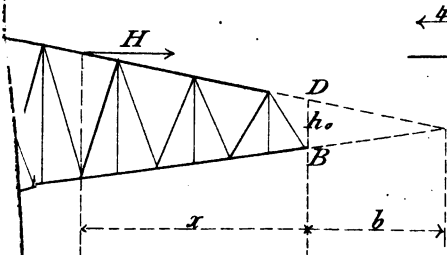


Fig. 16.



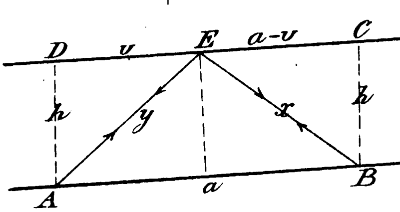


Fig. 18.

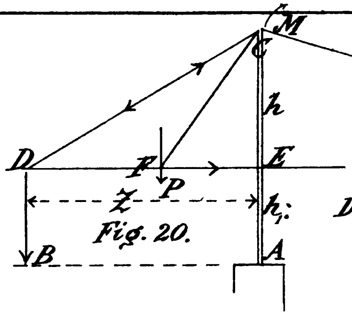


Fig. 20.

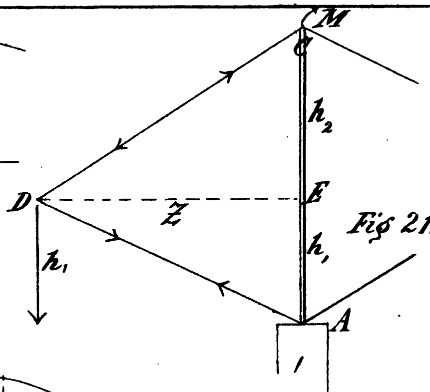
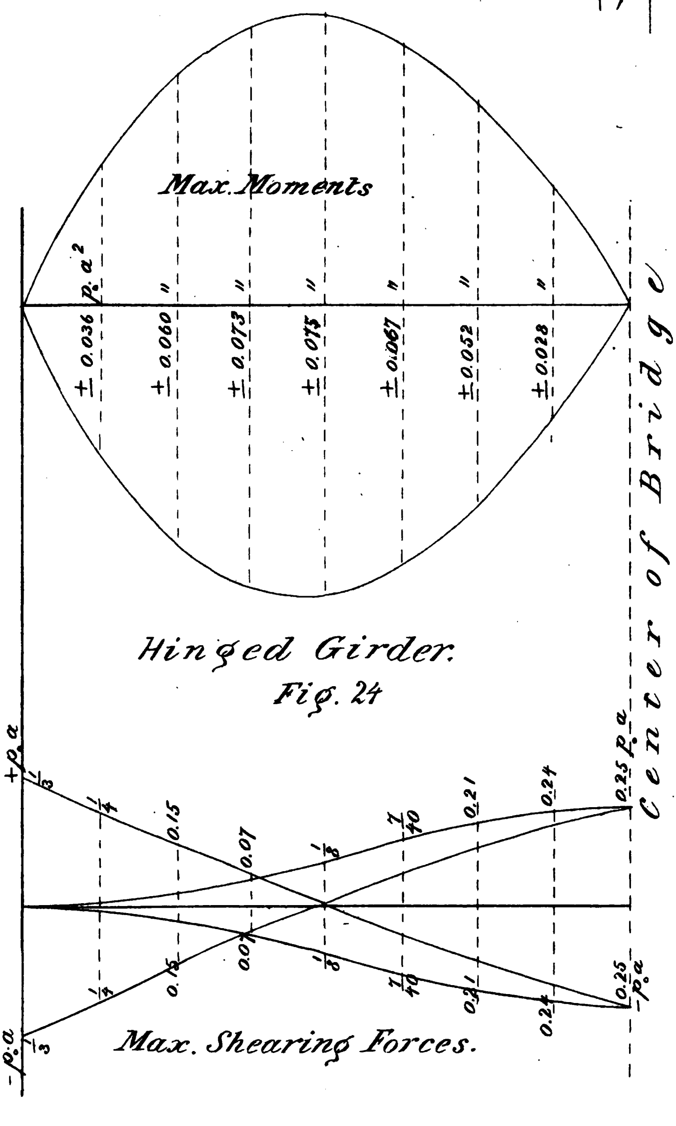


Fig 21.



Hinged Girder.
Fig. 24

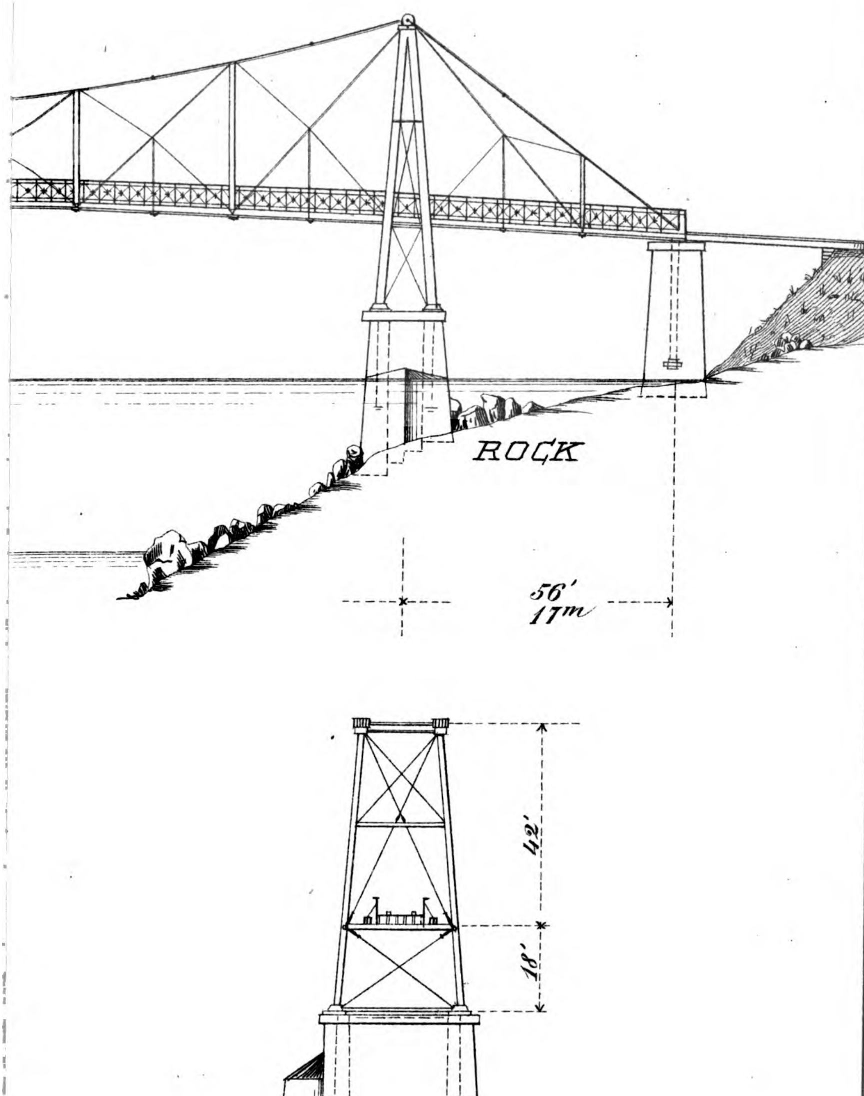
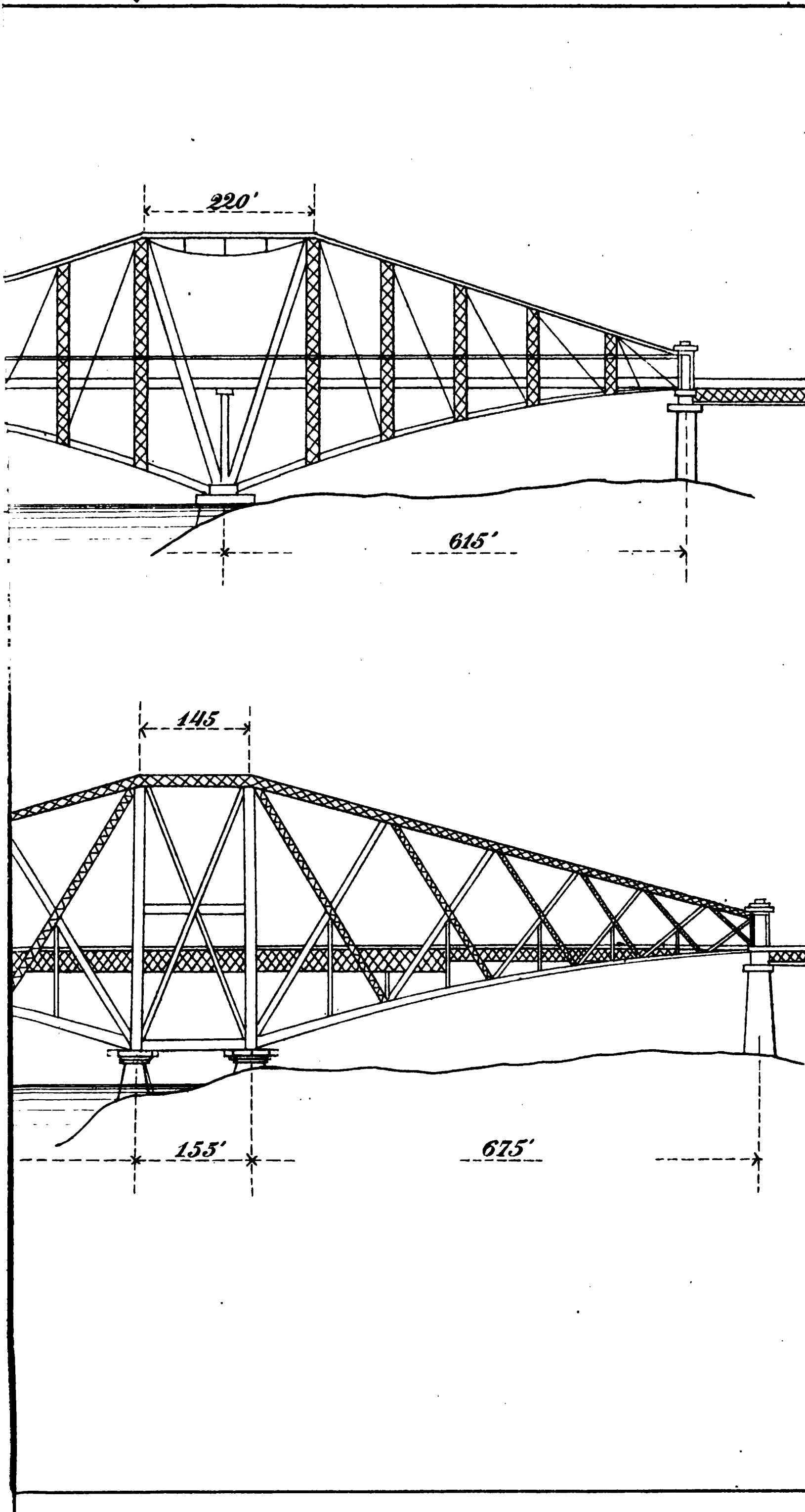

ROCK
56'
17m
42'
18'

Plate VI.



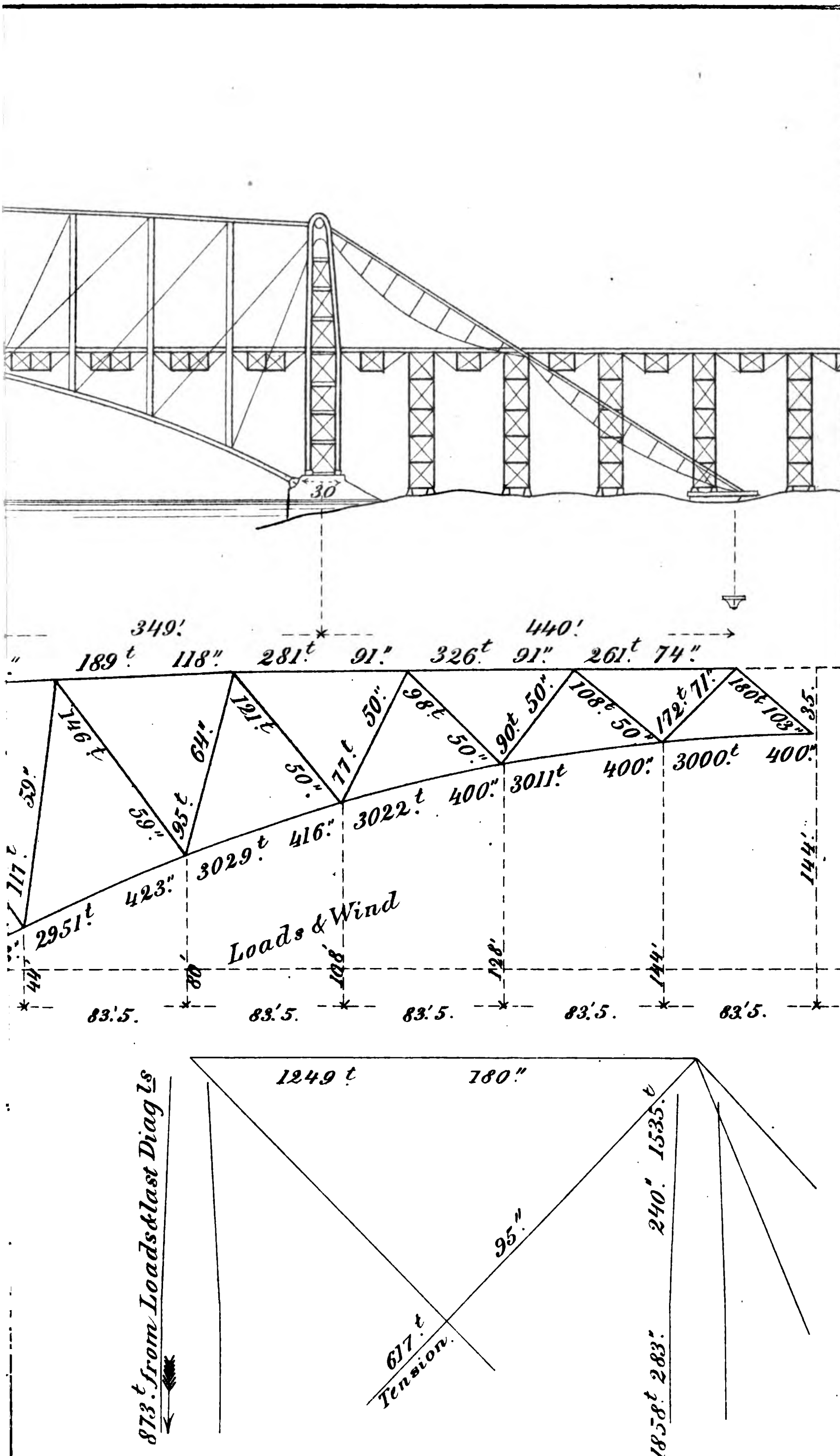
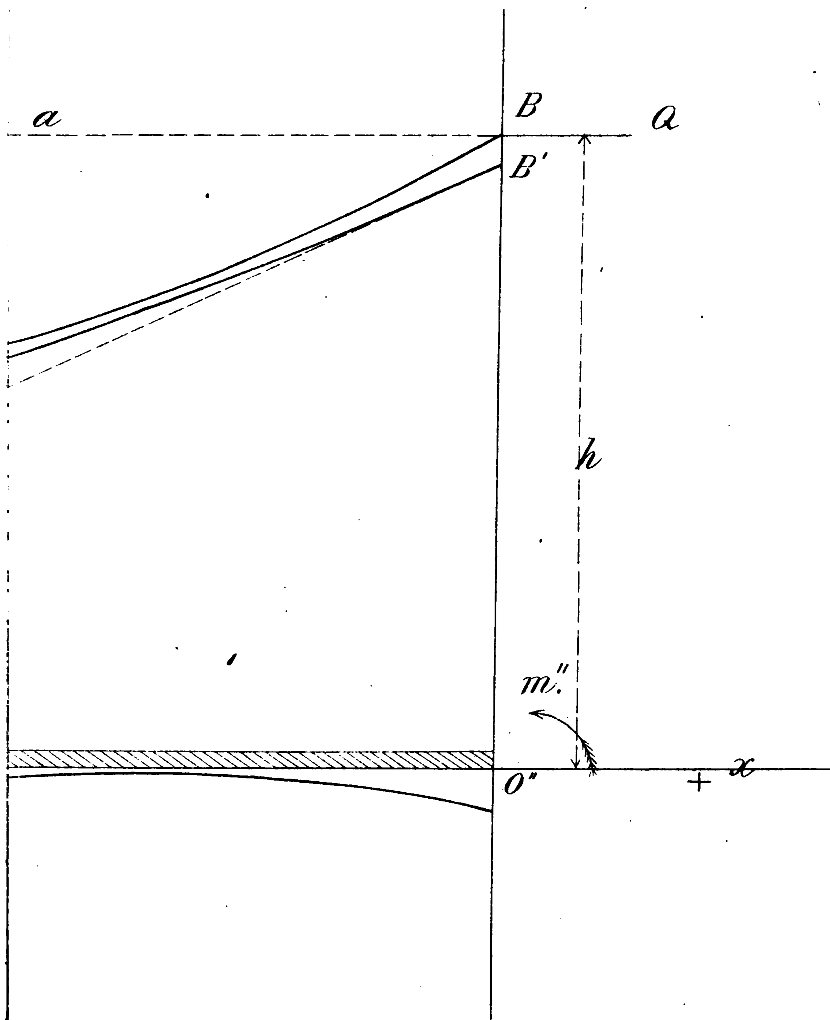

30
349'
440'
189 t
118"
281 t
91"
326 t
91"
261 t
74"
146 t
59"
121 t
64"
95 t
59"
50"
77 t
50"
98 t
50"
90 t
50"
108 t
50"
172 t
71"
180 t
103"
35'
117 t
59"
2951 t
423"
3029 t
416"
3022 t
400"
3011 t
400"
3000 t
400"
144'
Loads & Wind
44'
80'
108'
128'
144'
83'.5
83'.5
83'.5
83'.5
83'.5
83'5
1249 t
780"
1535 t
240"
95"
617 t
Tension
1858 t
283"
873 t from Loads & last Diag ls

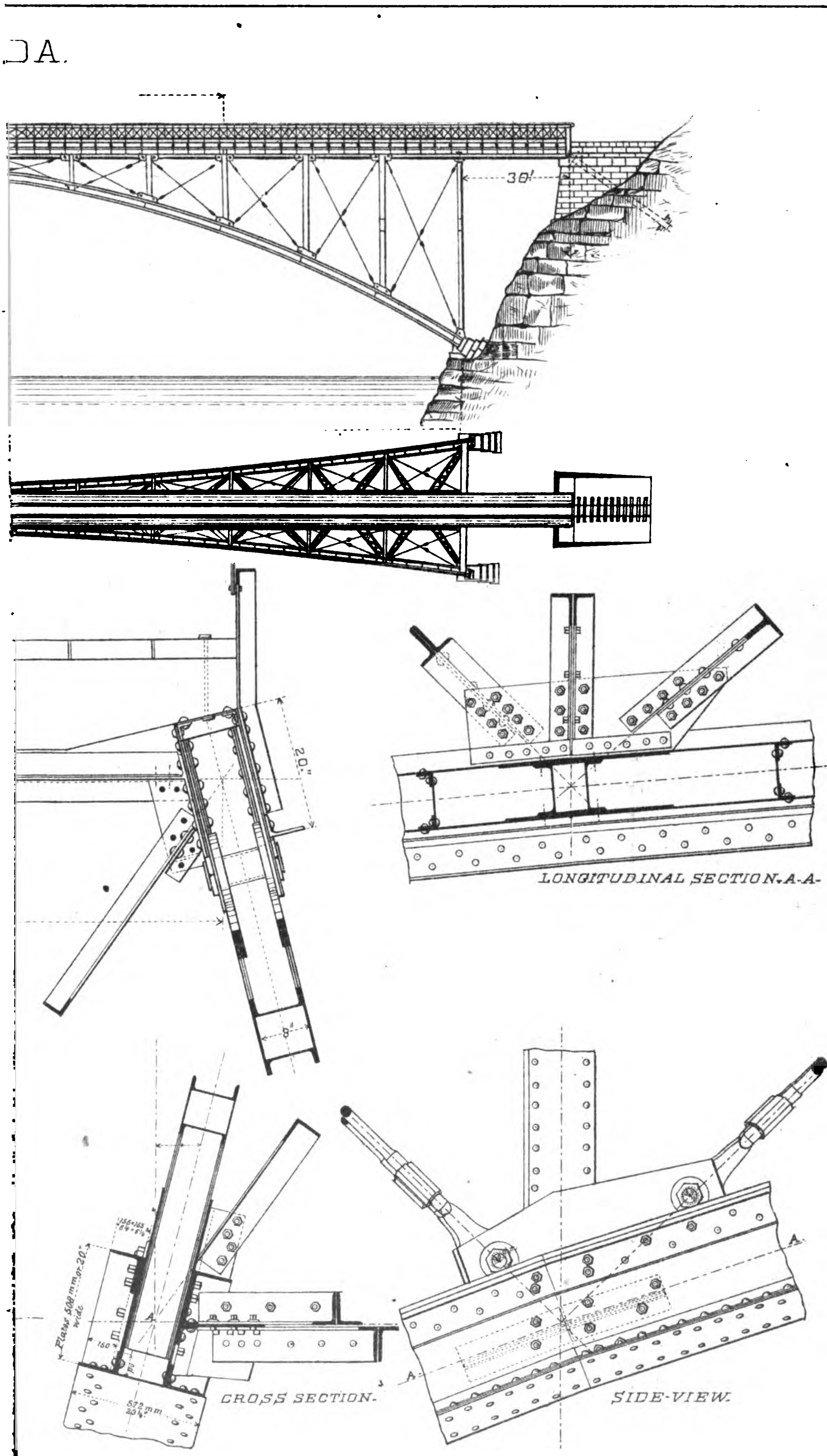
Plate VIII.



$+ \frac{H.x^2}{a^2}$

$) + \varphi(x).$

ity.



A.
30'
20"
8"
LONGITUDINAL SECTION-A-A-
Plates 508 mm or 20" wide
572 mm
CROSS SECTION-
SIDE-VIEW
A